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A CLASS OF TWO-BRIDGE KNOTS WITH PROPERTY-P

E. J. MAYLAND, JR.[1]

ABSTRACT. A knot k has *property*-P provided no simply connected manifold results from performing a nontrivial elementary surgery along k. We establish property-P for certain families of two-bridge knots generalizing twist knots (Whitehead doubles of the trivial knot).

Hamstrom and Jerrard [3] associate a two-bridge knot or link to each ordered pair of relatively prime integers (α, β) with $|\alpha| > |\beta|$. For each $p = 1, 2, \ldots$ we will consider the family of two-bridge knots associated with the pairs $(|2p(2m-1)+1|, 2p)$, $m = \pm 1, \pm 2, \ldots$. Schubert [10] had earlier used ordered pairs of odd, relatively prime integers to classify the associated knots, and in these terms we will consider the knots $(4pm - 2p + 1, 4p(m-1)+1)$, for $m > 0$, $p > 0$ and $(-4pm + 2p - 1, -4pm - 1)$ when $m < 0$ and $p > 0$. A knot k has *property*-P provided no simply connected manifold results from performing a nontrivial elementary surgery along k (see, e.g., [1]). We show that each knot in the union of the above families has property-P. This generalizes the result of Bing and Martin [1], González-Acuña [2], and, more recently, Riley [9], each of whom established the case $p = 1$ (yielding the family of *twist-knots*, or Whitehead doubles of the trivial knot). The class of knots we consider is indicated in Figure 1, and we remark that our result also extends that of Neuzil [7], who showed that when a regular neighborhood of a *nontrivial* knot J is replaced by the solid torus T of the figure, the image of K is a knot with property-P.

To begin the proof, we observe that the group of a two-bridge knot (α, β) can be computed directly from the characterising integers following [6] and [8].

We consider first the case $m > 0$. Taking the equivalent knots $(4pm - 2p + 1, 4p(m-1)+1)$, we find

$$G = \pi_1(S^3 - k) = \langle x_1, x_2 | W x_1 W^{-1} = x_2 \rangle \tag{1}$$

$$\text{where } W = \left((x_1^{-1}x_2)^{m-1} x_1^{-1} (x_2^{-1}x_1)^{m-1} x_2^{-1}\right)^p \text{ and } m, p > 0.$$

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In this group we will take x_1 and

$$L = x_1^{-4p} \cdot W^{-1} \cdot \left(\left(x_1 x_2^{-1} \right)^{m-1} x_1 \left(x_2 x_1^{-1} \right)^{m-1} x_2 \right)^p$$

as a (meridian, null-homologous longitude) pair on the boundary of a regular neighborhood of k. If we perform an elementary surgery along k by replacing this regular neighborhood by a solid torus T from S^3 in such a way that the meridian for T is identified with a simple closed curve in the class of $L^n x_1^{n_1}$, then the group $G(k\colon n_1, n)$ of the resulting manifold is obtained by adding the relation $L^n x_1^{n_1} = 1$ to the group (1). If $(n_1, n) \neq (\pm 1, 0)$, we say the surgery was *nontrivial*, and we will now show that the group $G(k\colon n_1, n)$ is nontrivial for all nontrivial surgeries.

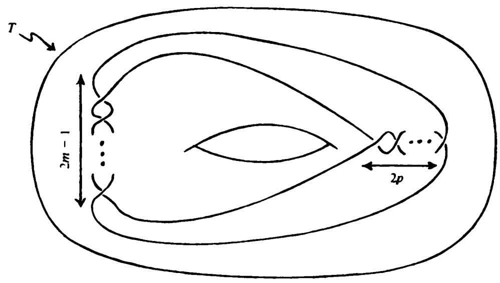

FIGURE 1a.
The two-bridge knots $(2p(2m-1)+1, 2p) = (4pm - 2p + 1, 4pm - 4p + 1)$ with $m > 0$ and $p > 0$.

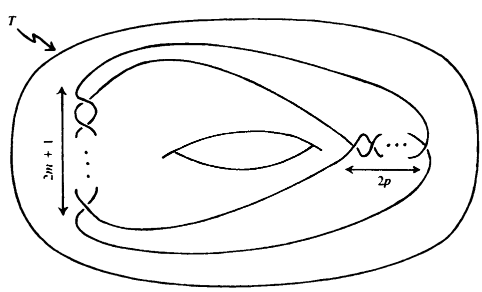

FIGURE 1b.
The two-bridge knots $(|2p(2m-1)+1|, 2p) = (-4pm + 2p - 1, -4pm - 1)$ with $m < 0$ and $p > 0$.

By homology considerations $G(k\colon n_1, n) \neq 1$ if $n_1 \neq \pm 1$. Thus by reversing orientations, if necessary, we can restrict ourselves to the case $n_1 = +1$. In $G(k\colon 1, n)$, we define $w = x_1^{-1} x_2$ and delete $x_2 = x_1 w$, yielding

$$G(k\colon 1, n) = \Big\langle x_1, w \,\Big|\, \left(w^{m-1} x_1^{-1} w^{-m} x_1^{-1} \right)^p \cdot x_1 \cdot \left(w^{m-1} x_1^{-1} w^{-m} x_1^{-1} \right)^{-p} = x_1 w,$$

$$\left[x_1^{-4p} \left(x_1 w^m x_1 w^{-m+1} \right)^p \cdot \left(x_1 w^{-m+1} x_1 w^m \right)^p \right]^n x_1 = 1 \Big\rangle \tag{2}$$

Defining $k = x_1 w^m$ and deleting $x_1 = kw^{-m}$ yields

$$(3)\quad \begin{aligned} G(k\colon 1, n) = \Big\langle k, w \,\big|\, &(w^{2m-1}k^{-2})^p \cdot kw^{-m} \cdot (w^{2m-1}k^{-2})^{-p} = kw^{-m+1}, \\ &\big[(kw^{-m})^{-4p} \cdot (k^2w^{-2m+1})^p \cdot (kw^{-2m+1}k)^p\big]^n kw^{-m} = 1\Big\rangle. \end{aligned}$$

To this we adjoin the relation $w^{2m-1} = 1$, so that $G(k\colon 1, n)$ is generated by k and w^m. Then, using the fact that L and x_1 lie on a common torus and must therefore have commuting images in any homomorphic image of the knot group, we send w^{-m} and k to A and B, mapping this quotient onto

$$(4)\qquad \overline{G}_n = \big\langle A, B \,|\, 1 = A^{2m-1} = B^{4p} = (AB)^{4pn-1} = (AB^{2p})^2\big\rangle$$

where $p, m > 0$ and $n \neq 0$.

We treat the case $m < 0$ analogously, temporarily replacing m by its absolute value. The two-bridge knots considered are therefore of the form $(4pm + 2p - 1, 2p)$ with $p > 0$ and $m > 0$. The groups of the equivalent knots $(4pm + 2p - 1, 4pm - 1)$ are given by

$$G = \pi_1(s^3 - k) = \langle x_1, x_2 \,|\, Wx_1W^{-1} = x_2\rangle$$

where $W = x_2^{-1}(x_2(x_1^{-1}x_2)^m x_1(x_2^{-1}x_1)^m)^p x_1^{-1}$ and $m, p > 0$.

After the corresponding sequence of Tietze transformations, we map the appropriate quotient of this group homomorphically onto the group

$$(4')\qquad \overline{G}_n = \big\langle A, B \,|\, 1 = A^{2m+1} = B^{4p} = (AB)^{4pn+1} = (AB^{2p})^2\big\rangle$$

where $p, m > 0$ and $n \neq 0$.

Apparently the presentations (4) and (4′) are identical if we allow positive and negative (nonzero) values of m, which amounts to abandoning our temporary convention of writing m for its absolute value in the (prime) case $m < 0$. It thus suffices for both cases to show the groups $\overline{G}_n$ of (4) are nontrivial for $p > 0$ and $m, n \neq 0$.

We have already remarked that the case $p = 1$ was settled by Bing and Martin, etc., and observe that the value $m = 1$ yields the $K(2p + 1, 2)$ torus knot with two-bridge form $(2p + 1, 1)$. Torus knots were shown to have property-P by Hempel [4].

We map (4) to the following group isomorphically by defining $C = AB^{2p}$ and deleting A:

$$(5)\quad \overline{G}_n = \big\langle B, C \,|\, 1 = B^{4p} = C^2 = (CB^{-2p+1})^{4pn-1} = (CB^{2p})^{2m-1}\big\rangle.$$

We consider this as the result of adjoining the relations $(CB^{-2p+1})^{4pn-1} = 1$ and $(CB^{2p})^{2m-1} = 1$ to the free product of cyclic groups of orders 2 and $4p$. Applying the theory of small cancellation over free products (see, e.g., [5]), we find the relations satisfy both triangle conditions and the small cancellation condition $C(d)$ for $d = \min\{|4pn - 1|, |2m - 1|\}$. Since $p > 1$ and $m \neq 0, 1$, this shows all the groups (5) are nontrivial unless $|2m - 1| = 3$.

The remaining case is

$$\bar{G}_n = \left\langle B, C \middle| 1 = B^{4p} = C^2 = (CB^{2p})^3 = (CB^{-2p+1})^{4pn-1} \right\rangle \tag{6}$$

where $p > 1$ and $n \neq 0$.

This group is generated by B^{2p-1} and C, and we can therefore present it as

$$\bar{G}_n = \left\langle C, D \middle| 1 = C^2 = D^{4p} = (CD^{2p})^3 = (CD)^{4pn-1} \right\rangle \tag{7}$$

where $p > 1$ and $n \neq 0$.

We will show these groups are nontrivial using permutation representations related to those of González-Acuña [2].

If $n = +1$, we map $\langle C, D | 1 = C^2 = D^{4p} = (CD^{2p})^3 = (CD)^{4p-1} \rangle$ into the alternating group A_{8p-2} via the homomorphism θ_1 determined by

$$C \mapsto (1, 4p + 1)(2, 6p + 1)(3, 6p + 2) \cdots (2p - 1, 8p - 2)(2p, 6p - 1),$$
$$D \mapsto (1, 2, \ldots, 4p)(4p + 1, 4p + 2, \ldots, 6p).$$

If $n > +1$, we observe the homomorphism θ_2 sending $\langle c, d | 1 = c^2 = d^{4p} = (cd^{2p})^3 = (cd)^{4p} \rangle$ into A_{8p} via

$$C \mapsto (1, 4p + 1)(2, 6p + 2)(3, 6p + 3) \cdots (2p - 1, 8p - 1)(2p, 6p),$$
$$D \mapsto (1, 2, \ldots, 4p)(4p + 1, \ldots, 6p)(6p + 1, 6p + 2)(8p - 1, 8p).$$

Then the above two maps can be combined to provide a homomorphism from (7) into the alternating group A_{8pn-2} via

$$\begin{aligned} C \mapsto \prod_{i=1}^{n-1} \big[&(8pi - 8p + 1, 8pi - 4p + 1)(8pi - 8p + 2, 8pi - 2p + 2) \\ &\ldots (8pi - 6p - 1, 8pi - 1)(8pi - 6p, 8pi - 2p)\big] \\ \times &(8pn - 8p + 1, 8pn - 4p + 1)(8pn - 8p + 2, 8pn - 2p + 1) \\ &\ldots (8pn - 6p - 1, 8pn - 2)(8pn - 6p, 8pn - 2p - 1), \end{aligned}$$

$$\begin{aligned} D \mapsto \prod_{i=1}^{n} &\big[(8pi - 8p + 1, \ldots, 8pi - 4p)\big] \\ \times \prod_{i=1}^{n-1} &\big[(8pi - 2p + 1, 8pi - 2p + 2)(8pi - 1, 8pi) \\ &\times (8pi - 4p + 1, 8pi + 4p + 2, \ldots, 8pi + 6p)\big] \\ \times &(8pn - 4p + 1, 4p + 2, \ldots, 6p). \end{aligned}$$

The case $n < 0$ is completely analogous. If $n = -1$ we map the group

$\langle C, D \mid 1 = C^2 = D^{4p} = (CD^{2p})^3 = (CD)^{4p+1} \rangle$ into the alternating group A_{8p+2} via a homomorphism θ_3 defined by

$$C \mapsto (1, 4p + 1)(2, 6p + 2)(3, 6p + 3) \cdots (2p - 2, 8p - 2)(8p + 2, 4p - 1)(2p, 4p),$$

$$D \mapsto (1, \ldots, 4p)(4p + 1, \ldots, 6p)(6p + 1, \ldots, 8p)(8p + 1, 8p + 2).$$

If $n < -1$, the homomorphisms θ_3 and θ_2 are combined just as in the case $n > +1$. This shows the nontriviality of the groups (5) and completes the proof.

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Department of Mathematics, York University, Downsview, Ontario M3J 1P3, Canada