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A NOTE ON THE CENTRAL LIMIT THEOREM FOR SQUARE-INTEGRABLE PROCESSES

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ABSTRACT. A method is given for constructing sample-continuous processes which do not satisfy the central limit theorem in $C[0, 1]$. Let $\{X(t): t \in [0, 1]\}$ be a stochastic process. Using our method we characterize all possible nonnegative functions f for which the condition

$$E(X(t) - X(s))^2 < f(|t - s|)$$

alone is sufficient to imply that $X(t)$ satisfies the central limit theorem in $C[0, 1]$.

1. **Introduction.** Let $C = C[0, 1]$ denote the space of real-valued continuous functions on the unit interval. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent C-valued random variables with the same distribution, $\mathcal{L}(X)$. Assume that they are defined on the same probability space $(\Omega, \mathfrak{F}, \Pr)$ and that for $t \in [0, 1]$, $EX(t) = 0$ and $EX^2(t) < \infty$. Let $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$. X or $\mathcal{L}(X)$ is said to *satisfy the central limit theorem* (CLT) *in* C if there exists a sample-continuous Gaussian process Z such that for one and hence all sequences $\{X_i\}$ as above, $\mathcal{L}(Z_n) \to \mathcal{L}(Z)$ weakly in C; i.e., for every bounded continuous real function g on C, $Eg(Z_n) \to Eg(Z)$. Z is called the limiting Gaussian process.

In this note we give a method for constructing sample-continuous processes which do *not* satisfy the CLT in C. A first application of this method appears in Hahn (1977).

Using this method we will show that when the only known information about a process $X(t)$ is of the form

(1.1) for some $\varepsilon > 0$ and some nonnegative function f on [0, 1] which is nondecreasing on $[0, \varepsilon]$,

$$E(X(t) - X(s))^2 \leqslant f(|t - s|), \qquad |t - s| \leqslant \varepsilon,$$

then the best possible sufficient condition for X to satisfy the CLT in C is

$$\int_0 y^{-3/2} \mathfrak{f}^{1/2}(y)\, dy < \infty, \tag{1.2}$$

where

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$$\mathbf{f}(s)=\begin{cases}\inf_{y\geqslant 1} y^2 f(s/y) & \text{if } s\in[0,\varepsilon],\\ f(s) & \text{if } s>\varepsilon.\end{cases}$$

In Hahn and Klass (1977) it was shown that under assumption (1.1), the best possible condition for determining sample-continuity is (1.2).

2. **Method for constructing sample-continuous processes which do not satisfy the CLT in** C. Let $\{X(t), t\in[0,1]\}$ be a stochastic process on a probability space $(\Omega,\mathcal{F},P)$ which prossesses the following properties:

$$EX^2(t)<\infty \quad \text{for all } t\in[0,1]. \tag{2.1}$$

There is a set $A\in\mathcal{F}$ with $P(A)=\delta>0$ which contains a decreasing sequence of sets $A_n\in\mathcal{F}$ with $A\supset A_1$, $P(A_n)>0$ for all n and $\lim_{n\to\infty}P(A_n)=0$ such that

(2.2) for each $\omega\in A$ there is a nonempty subset $T(\omega)\subset[0,1]$ with the property that if $t\in T(\omega)$ then

$$\lim_{s\downarrow t} X(s,\omega)=\lim_{r\uparrow t} X(r,\omega)=\pm\infty;$$

(2.3) for each $\omega\in A$, $X(t,\omega)$ is continuous on $[0,1]\sim T(\omega)$;

(2.4) for $\omega\in A^c$, $X(t,\omega)$ is continuous on $[0,1]$.

It is easy to see that such processes exist. A few examples when $\Omega=[0,1]$ with Lebesgue measure, $A=[0,1]$ and $T(\omega)=\omega$ are

$$X(t,\omega)=\begin{cases}\text{either } |t-\omega|^{-1/4} \text{ or } \log|t-\omega| & \text{if } t\neq\omega,\\ 0 & \text{if } t=\omega.\end{cases}$$

A stochastic process X with the above properties is not sample-continuous. However, as we will now show, it can be modified in such a way that it is both sample-continuous and does not satisfy the CLT in C.

We begin by choosing a function R from Ω to $[0,\infty)$ for which

$$\lim_{n\to\infty} nP\{\omega\in A\colon R(\omega)\geqslant\sqrt{n}\}=\infty. \tag{2.5}$$

To see that such a function exists, let A_n be the decreasing sequence of sets contained in A. Let $a_n=P(A_n)$. Extract a decreasing subsequence a_{n_k} with the property that $k^2a_{n_{k-1}}\to\infty$. Let $R^2(\omega)=\inf\{k\colon \omega\notin A_{n_k}\}$. Then $\{\omega\in A\colon R^2(\omega)\geqslant k\}=A_{n_{k-1}}$, so $R(\omega)$ satisfies (2.5).

The desired modification of $X(t,\omega)$ is now obtained by first letting

$$\tilde{X}(t,\omega)=\begin{cases}(\operatorname{sgn}X(t,\omega))(|X(t,\omega)|\wedge R(\omega)) & \text{if } t\notin T(\omega),\\ (\operatorname{sgn}X(t,\omega))R(\omega) & \text{if } t\in T(\omega),\end{cases}$$

and finally symmetrizing to yield

$$Y(t,\omega)=Y(t,\omega\times k)=\begin{cases}\tilde{X}(t,\omega) & \text{if } k=0,\\ -\tilde{X}(t,\omega) & \text{if } k=1\end{cases}$$

on the space $(\Omega \times \{0,1\}, \mathbf{P})$ where $\mathbf{P} \equiv P \times (\delta_0/2 + \delta_1/2)$.

THEOREM 1. *The sample-continuous process $Y(t,\omega)$ does not satisfy the CLT in C.*

PROOF. Let $Y^{(i)}(t), i = 1, 2, \ldots$ denote i.i.d. copies of $Y(t)$, $S_n(t) = \sum_{i=1}^n Y^{(i)}(t)$ and $Z_n(t) = S_n(t)/\sqrt{n}$. We can assume that the independent copies of Y are taken on a product space, $Y^{(i)}(t,\omega) = Y(t, \omega(i))$ where $\omega(i) = \omega(i) \times j, j = 0$ or 1.

In order to show that $Y(t)$ does not satisfy the CLT it suffices to show that $\{Z_n\}$ is not uniformly bounded in probability, i.e., there exists $\varepsilon > 0$ such that for $b > 0$ there is an $n(b)$ for which $\mathbf{P}\{\sup_t |Z_{n(b)}(t)| \geqslant b\} > \varepsilon$.

We begin by showing that for any $b > 0$, there exists N_b such that $n \geqslant N_b$ implies that

$$\mathbf{P}\Big\{ \max_{1 \leqslant i \leqslant n} \sup_t |Y^{(i)}(t)|/\sqrt{n} \geqslant 2b \Big\} > \frac{1}{2}.$$

Since $\sup_t |Y^{(i)}(t,\omega)|/\sqrt{n} = R(\omega(i))/\sqrt{n}$,

$$\begin{aligned}
\mathbf{P}\Big\{\omega \in \Omega \times \{0,1\}: \max_{1 \leqslant i \leqslant n} \sup_t |Y^{(i)}(t,\omega)|/\sqrt{n} < 2b\Big\} \\
&= P^n\Big\{\tilde{\omega} \in \Omega^n: \max_{1 \leqslant i \leqslant n} R(\omega(i))/\sqrt{n} < 2b\Big\} \\
&= (P\{\omega \in \Omega: R(\omega)/\sqrt{n} < 2b\})^n \quad \text{by independence} \\
&\leqslant \exp(-nP\{\omega \in \Omega: R(\omega) \geqslant 2b\sqrt{n}\}) \to 0 \quad \text{as } n \to \infty \text{ by (2.5)}.
\end{aligned}$$

Thus, there exists N_b such that $n \geqslant N_b$ implies

$$\mathbf{P}\Big\{ \max_{1 \leqslant i \leqslant n} \sup_t |Y^{(i)}(t)|/\sqrt{n} < 2b \Big\} < \frac{1}{2};$$

and hence,

$$\mathbf{P}\Big\{ \max_{1 \leqslant i \leqslant n} \sup_t |Y^{(i)}(t)|/\sqrt{n} \geqslant 2b \Big\} > \frac{1}{2}.$$

Consequently, letting $\varepsilon = \frac{1}{4}$, if $n \geqslant N_b$,

$$\mathbf{P}\Big\{ \max_{1 \leqslant i \leqslant n} \sup_t |S_i(t)|/\sqrt{n} \geqslant b \Big\} > 2\varepsilon.$$

Applying the Lévy inequality for processes (see Dudley (1967), Lemma 4.4, p. 300 or Kahane (1968), Lemma 1, p. 12), we see that if $n \geqslant N_b$ then

$$\mathbf{P}\Big\{ \sup_t |Z_n(t)| \geqslant b \Big\} \geqslant \frac{1}{2}\mathbf{P}\Big\{ \max_{1 \leqslant i \leqslant n} \sup_t |S_i(t)|/\sqrt{n} \geqslant b \Big\} > \varepsilon. \quad \square$$

3. **Moment conditions on increments and the CLT.** Let $\{X(t), t \in [0,1]\}$ be a stochastic process satisfying properties (1.1) and (1.2). As shown in Hahn and Klass (1977), proof of Theorem 1, $E(X(t) - X(s))^2 \leqslant 4\mathbf{f}(|t-s|)$. Consequently, by Theorem 2.5 of Hahn (1977), condition (1.2) is sufficient for X to satisfy the CLT in C. The following theorem shows that this result is best possible.

THEOREM 2. *If f is a nonnegative function which is nondecreasing on $[0, \varepsilon]$ and such that $\int_0 y^{-3/2}\mathbf{f}^{1/2}(y)\,dy = \infty$, there exists a sample-continuous process $Y(t, \omega)$ which does not satisfy the CLT in C but such that*

$$E(Y(t) - Y(s))^2 \leqslant f(|t-s|), \qquad |t-s| \leqslant \varepsilon.$$

PROOF. In §4 of Hahn and Klass (1976) a real-valued stochastic process, $X(t, \omega)$, was constructed on $[0,1] \times ([0,1]$, Lebesgue) such that for each t,

$$X(t,\omega) = \begin{cases} (2\sqrt{2}\pi)^{-1} \sum_{k \geqslant 1} b_k \cos 2\pi k(t-\omega), & 0 < |t-\omega| < 1, \\ 0 & |t-\omega| = 0 \text{ or } 1, \end{cases}$$

where the sequence $\{b_k\}$ has the following properties:

(1) $b_k \geqslant b_{k+1}$;
(2) $\sum_{k \geqslant 1} b_k = \infty$;
(3) $\sum_{k=1}^{j} k^2 b_k^2 + j^2 \sum_{k \geqslant j+1} b_k^2 \leqslant j^2 \mathbf{f}(1/j)$;
(4) kb_k is bounded.

For this process $E(X(t) - X(s))^2 \leqslant f(|t-s|)$. Since the sequence $\{b_k\}$ decreases, $X(t, \omega)$ is continuous for $0 < |t-\omega| < 1$. As shown in Lemma 3 of Hahn and Klass, conditions (1), (2), and (4) imply that $\lim_{t \to \omega} X(t, \omega) = \infty$. Since $\cos 2\pi kx = \cos 2\pi k(1-x)$, the same argument shows that for fixed $\omega = 0$, or 1, $\lim_{|t-\omega| \to 1} X(t, \omega) = \infty$.

Let $R(\omega) = (1-\omega)^{-1}$. $R(\omega)$ satisfies (2.5). The desired sample-continuous process $Y(t, \omega)$ on $([0,1] \times \{0,1\}$, Lebesgue $\times (\delta_0/2 + \delta_1/2) \equiv \lambda \times \mu)$ may be derived from $X(t, \omega)$ by the method given in §2. Theorem 1 now shows that Y does not satisfy the CLT in C.

Furthermore,

$$E_{\lambda \times \mu}(Y(t) - Y(s))^2 = E_\lambda(\tilde{X}(t) - \tilde{X}(u))^2 \leqslant E(X(t) - X(s))^2$$
$$\leqslant f(|t-s|), \qquad |t-s| \leqslant \varepsilon. \quad \square$$

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