A NOTE ON THE CENTRAL LIMIT THEOREM
FOR SQUARE-INTEGRABLE PROCESSES

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Abstract. A method is given for constructing sample-continuous processes which do not satisfy the central limit theorem in $C[0,1]$. Let $\{X(t): t \in [0,1]\}$ be a stochastic process. Using our method we characterize all possible nonnegative functions $f$ for which the condition

$$E(X(t) - X(s))^2 < f(|t - s|)$$

alone is sufficient to imply that $X(t)$ satisfies the central limit theorem in $C[0,1]$.

1. Introduction. Let $C = C[0,1]$ denote the space of real-valued continuous functions on the unit interval. Let $\{X_n, n \geq 1\}$ be a sequence of independent $C$-valued random variables with the same distribution, $\mathcal{E}(X)$. Assume that they are defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and that for $t \in [0,1]$, $EX(t) = 0$ and $EX^2(t) < \infty$. Let $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$. $X$ or $\mathcal{E}(X)$ is said to satisfy the central limit theorem (CLT) in $C$ if there exists a sample-continuous Gaussian process $Z$ such that for one and hence all sequences $\{X_i\}$ as above, $\mathcal{E}(Z_n) \rightarrow \mathcal{E}(Z)$ weakly in $C$; i.e., for every bounded continuous real function $g$ on $C$, $ Eg(Z_n) \rightarrow Eg(Z)$. $Z$ is called the limiting Gaussian process.

In this note we give a method for constructing sample-continuous processes which do not satisfy the CLT in $C$. A first application of this method appears in Hahn (1977).

Using this method we will show that when the only known information about a process $X(t)$ is of the form

$$E(X(t) - X(s))^2 < f(|t - s|), \quad |t - s| < \varepsilon,$$

(1.1) for some $\varepsilon > 0$ and some nonnegative function $f$ on $[0, 1]$ which is nondecreasing on $[0, \varepsilon]$

then the best possible sufficient condition for $X$ to satisfy the CLT in $C$ is

$$\int_0^\infty y^{-3/2} f^{1/2}(y) dy < \infty,$$

(1.2)

where

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In Hahn and Klass (1977) it was shown that under assumption (1.1), the best possible condition for determining sample-continuity is (1.2).

2. Method for constructing sample-continuous processes which do not satisfy the CLT in $C$. Let $\{X(t), t \in [0,1]\}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ which possesses the following properties:

$$(2.1) \quad EX^2(t) < \infty \text{ for all } t \in [0,1].$$

There is a set $A \in \mathcal{F}$ with $P(A) = \delta > 0$ which contains a decreasing sequence of sets $A_n \in \mathcal{F}$ with $A \supset A_1$, $P(A_n) > 0$ for all $n$ and $\lim_{n\to\infty} P(A_n) = 0$ such that

$$(2.2) \text{ for each } \omega \in A \text{ there is a nonempty subset } T(\omega) \subset [0,1] \text{ with the property that if } t \in T(\omega) \text{ then}$$

$$\lim_{s \uparrow t} X(s, \omega) = \lim_{r \downarrow t} X(r, \omega) = \pm \infty;$$

$$(2.3) \text{ for each } \omega \in A, X(t, \omega) \text{ is continuous on } [0,1] \sim T(\omega);$$

$$(2.4) \text{ for } \omega \in A^c, X(t, \omega) \text{ is continuous on } [0,1].$$

It is easy to see that such processes exist. A few examples when $\Omega = [0,1]$ with Lebesgue measure, $A = [0,1]$ and $T(\omega) = \omega$ are

$$X(t, \omega) = \begin{cases} \text{either } |t - \omega|^{-1/4} \text{ or } \log |t - \omega| & \text{if } t \neq \omega, \\ 0 & \text{if } t = \omega. \end{cases}$$

A stochastic process $X$ with the above properties is not sample-continuous. However, as we will now show, it can be modified in such a way that it is both sample-continuous and does not satisfy the CLT in $C$.

We begin by choosing a function $R$ from $\Omega$ to $[0, \infty)$ for which

$$(2.5) \quad \lim_{n \to \infty} nP(\omega \in A: R(\omega) \geq \sqrt{n}) = \infty.$$ 

To see that such a function exists, let $A_n$ be the decreasing sequence of sets contained in $A$. Let $a_n = P(A_n)$. Extract a decreasing subsequence $a_{n_k}$ with the property that $k^2 a_{n_{k-1}} \to \infty$. Let $R^2(\omega) = \inf\{k: \omega \notin A_{n_k}\}$. Then $\{\omega \in A: R^2(\omega) \geq k\} = A_{n_{k-1}}$, so $R(\omega)$ satisfies (2.5).

The desired modification of $X(t, \omega)$ is now obtained by first letting

$$\bar{X}(t, \omega) = \begin{cases} (\text{sgn } X(t, \omega))(|X(t, \omega)| \wedge R(\omega)) & \text{if } t \notin T(\omega), \\ (\text{sgn } X(t, \omega))R(\omega) & \text{if } t \in T(\omega), \end{cases}$$

and finally symmetrizing to yield

$$Y(t, \omega) = Y(t, \omega \times k) = \begin{cases} \bar{X}(t, \omega) & \text{if } k = 0, \\ -\bar{X}(t, \omega) & \text{if } k = 1 \end{cases}$$
on the space \((\Omega \times \{0,1\}, P)\) where \(P = P \times (\delta_0/2 + \delta_1/2)\).

**Theorem 1.** The sample-continuous process \(Y(t, \omega)\) does not satisfy the CLT in \(C\).

**Proof.** Let \(Y^{(i)}(t), i = 1, 2, \ldots\) denote i.i.d. copies of \(Y(t), S_n(t) = \sum_{i=1}^n Y^{(i)}(t)\) and \(Z_n(t) = S_n(t)/\sqrt{n}\). We can assume that the independent copies of \(Y\) are taken on a product space, \(Y^{(i)}(t, \omega) = Y(t, \omega(i))\) where \(\omega(i) = \omega(i) \times j, j = 0\) or 1.

In order to show that \(Y(t)\) does not satisfy the CLT it suffices to show that \(\{Z_n\}\) is not uniformly bounded in probability, i.e., there exists \(\epsilon > 0\) such that for \(b > 0\) there is an \(n(b)\) for which \(P\{\sup_t|Z_n(t)| > b\} > \epsilon\).

We begin by showing that for any \(b > 0\), there exists \(N_b\) such that \(n > N_b\) implies that

\[
P\left\{\max_{1 \leq i \leq n} \sup_t \left|Y^{(i)}(t)\right| / \sqrt{n} > 2b\right\} > \frac{1}{2}.
\]

Since \(\sup_t|Y^{(i)}(t, \omega)| / \sqrt{n} = R(\omega(i)) / \sqrt{n}\),

\[
P\left\{\omega \in \Omega \times \{0,1\}: \max_{1 \leq i \leq n} \sup_t |Y^{(i)}(t, \omega)| / \sqrt{n} < 2b\right\}
\]

\[
= P^n\left\{\bar{\omega} \in \Omega^n: \max_{1 \leq i \leq n} R(\omega(i)) / \sqrt{n} < 2b\right\}
\]

\[
= (P(\omega \in \Omega: R(\omega) / \sqrt{n} < 2b))^n \text{ by independence}
\]

\[
\leq \exp(-nP(\omega \in \Omega: R(\omega) > 2b\sqrt{n})) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ by (2.5)}.
\]

Thus, there exists \(N_b\) such that \(n > N_b\) implies

\[
P\left\{\max_{1 \leq i \leq n} \sup_t |Y^{(i)}(t)| / \sqrt{n} < 2b\right\} < \frac{1}{2};
\]

and hence,

\[
P\left\{\max_{1 \leq i \leq n} \sup_t |Y^{(i)}(t)| / \sqrt{n} > 2b\right\} > \frac{1}{2}.
\]

Consequently, letting \(\epsilon = \frac{1}{4}\), if \(n > N_b\),

\[
P\left\{\max_{1 \leq i \leq n} \sup_t |S_n(t)| / \sqrt{n} > b\right\} > 2\epsilon.
\]

Applying the Lévy inequality for processes (see Dudley (1967), Lemma 4.4, p. 300 or Kahane (1968), Lemma 1, p. 12), we see that if \(n > N_b\) then

\[
P\left\{\sup_t |Z_n(t)| > b\right\} > \frac{1}{2}P\left\{\max_{1 \leq i \leq n} \sup_t |S_n(t)| / \sqrt{n} > b\right\} > \epsilon. \quad \square
\]
3. Moment conditions on increments and the CLT. Let \( \{X(t), t \in [0,1]\} \) be a stochastic process satisfying properties (1.1) and (1.2). As shown in Hahn and Klass (1977), proof of Theorem 1, \( E(X(t) - X(s))^2 \leq 4\Phi(|t - s|) \). Consequently, by Theorem 2.5 of Hahn (1977), condition (1.2) is sufficient for \( X \) to satisfy the CLT in \( C \). The following theorem shows that this result is best possible.

**Theorem 2.** If \( f \) is a nonnegative function which is nondecreasing on \([0, \varepsilon] \) and such that \( \int_0^\varepsilon y^{-3/2} f^{1/2}(y) \, dy = \infty \), there exists a sample-continuous process \( Y(t, \omega) \) which does not satisfy the CLT in \( C \) but such that

\[
E(Y(t) - Y(s))^2 < f(|t - s|), \quad |t - s| < \varepsilon.
\]

**Proof.** In §4 of Hahn and Klass (1976) a real-valued stochastic process, \( X(t, \omega) \), was constructed on \([0,1] \times ([0,1], \text{Lebesgue}) \) such that for each \( t \),

\[
X(t, \omega) = \begin{cases} 
(2\sqrt{2}\pi)^{-1} \sum_{k \geq 1} b_k \cos 2\pi k (t - \omega), & 0 < |t - \omega| < 1, \\
0 & |t - \omega| = 0 \text{ or } 1,
\end{cases}
\]

where the sequence \( \{b_k\} \) has the following properties:

1. \( b_k \geq b_{k+1} \);
2. \( \sum_{k \geq 1} b_k = \infty \);
3. \( \sum_{k=1}^\infty b_k b_{k+j} / j^2 \sum_{k \geq j+1} b_k^2 < j^2 f(1/j) \);
4. \( \{b_k\} \) is bounded.

For this process \( E(X(t) - X(s))^2 \leq f(|t - s|) \). Since the sequence \( \{b_k\} \) decreases, \( X(t, \omega) \) is continuous for \( 0 < |t - \omega| < 1 \). As shown in Lemma 3 of Hahn and Klass, conditions (1), (2), and (4) imply that \( \lim_{|t-\omega| \to 1^-} X(t, \omega) = \infty \).

Since \( \cos 2\pi kx = \cos 2\pi k(1 - x) \), the same argument shows that for fixed \( \omega = 0 \) or \( 1 \), \( \lim_{|t-\omega| \to 1^-} X(t, \omega) = \infty \).

Let \( R(\omega) = (1 - \omega) \cdot R(\omega) \) satisfies (2.5). The desired sample-continuous process \( Y(t, \omega) \) on \([0,1] \times \{0,1\}, \text{Lebesgue} \times (\delta_0/2 + \delta_1/2) = \lambda \times \mu \) may be derived from \( X(t, \omega) \) by the method given in §2. Theorem 1 now shows that \( Y \) does not satisfy the CLT in \( C \).

Furthermore,

\[
E_{\lambda \times \mu} (Y(t) - Y(s))^2 = E_{\lambda} (\tilde{X}(t) - \tilde{X}(\omega))^2 \leq E(X(t) - X(s))^2 \leq f(|t - s|), \quad |t - s| < \varepsilon.
\]

**References**


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