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POWER MAPS AND PRINCIPAL BUNDLES

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ABSTRACT. Let G be a path connected topological group. We investigate the integers m for which the mth power map on G extends to an overmap of principal G-bundles.

This, our third note on the subject of fibre-preserving maps (called overmaps) succeeds [5], [6]. I wish to thank Professor I. M. James who supervised [4], for encouragement, and for considerable help with the exposition.

1. **Statement of results.** Let G be a path connected topological group, and let P, P' be principal G-bundles over a path connected pointed base (B, b). An overmap $P \to P'$ *is of power* m when its restriction to fibres over b is the mth power map. By subtracting overmaps we see that there is an integer $\alpha(P) \geqslant 0$ *such that* there is an overmap $P \to B \times G$ of power m if and only if m is divisible by $\alpha(P)$. Evidently P is trivial if and only if $\alpha(P) = 1$.

THEOREM 1. *Suppose that there is an overmap $P \to P'$ of power n. Then there is an overmap $P \to P'$ of power m if and only if $m \equiv n \bmod \alpha(P)$.*

By means of the G-map k'_m: $G \times G \to G$ given by $k'_m(g, h) = h(g^{-1}h)^{m-1}$, the arguments of [5] prove a version of Theorem 1. However, it is better to prove Theorem 1 as follows.

Denote the reverse G-action [8, 8.11] on P' by

$$j\colon P' \times_B (B \times G) = P' \times G \to P',$$

and let $f\colon P \to P'$ be an overmap of power n. Given an overmap $e\colon P \to B \times G$, let $e^+\colon P \to P'$ be $f \times_B e\colon P \to P' \times_B (B \times G)$ followed by j.

Let k be inverse to the bundle equivalence

$$p_1 \times_B j\colon P' \times_B (B \times G) \to P' \times_B P'$$

where p_i means projection to the ith factor. Given an overmap $d\colon P \to P'$, let $d^-\colon P \to B \times G$ be $f \times_B d\colon P \to P' \times_B P'$ followed by $p_2 k$. Since $e^{+-} = e$, $d^{-+} = d$, we have a bijection between overmaps $P \to B \times G$ of power m and overmaps $P \to P'$ of power $m + n$. This proves Theorem 1. From now on we suppose that B, G are finite CW-complexes.

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THEOREM 2. *Suppose that there is an overmap $P' \to P$ of power n. Then there is an overmap $P \to P'$ of power m if and only if*

$$mn \equiv 1 \mod \text{l.c.m.}(\alpha(P),\ \alpha(P')).$$

For example let P be induced from the Hopf bundle $S^7 \to S^4$ (which we regard as a principal S^3-bundle) by the map $S^3 \cup_r e^4 \to S^4$ collapsing S^3. By [4, II.4.2], $\alpha(P)$ is $|r|$ or $2|r|$, and if r is odd or if r is divisible by 8 then $\alpha(P) = |r|$.

Let H be another connected topological group that is a finite CW-complex, and let Q be a principal H-bundle over B.

THEOREM 3. *The primes that divide $\alpha(P \times_B Q)$ are precisely those that divide $\alpha(P)$ or $\alpha(Q)$.*

THEOREM 4. *We have $\alpha(P)$ positive if and only if the rational cohomology characteristic classes of P are zero.*

For example if B is a sphere of odd dimension, or a real projective space, or a generalised lens space, then $\alpha(P)$ is positive. Hence in these cases there are overmaps $P \to P$ of powers greater than 1. However if P is associated with the tangent bundle to S^{2q} $(q > 0)$ then $\alpha(P) = 0$. In this case all overmaps $P \to P$ that restrict to power maps are of power 1.

2. **Function space bundles.** We work in the category of compactly generated spaces [9]. Given $f\colon F \to F'$, let $\mathcal{G}_f$ be the space of maps homotopic to f. Let G' be a path connected topological group and let F, F' be G'-spaces. Then a G'-action '$*$' on $\mathcal{G}_f$ is given by $(g * e)(x) = g \cdot (e(g^{-1} \cdot x))$. Here $g \in G'$, $e \in \mathcal{G}_f$, $x \in F$.

Let R be a principal G'-bundle over a path connected pointed base (B, b). Formation of associated bundles defines a functor (which we also denote by R) from the category of G'-spaces and G'-maps to the category of compactly generated overspaces of B and overmaps.

Given $q\colon Y \to B$ with nonempty fibre Z, let $\mathfrak{M}Y$ be the space maps $F \to Y$ whose composites with q are constant. Given $f\colon F \to Z$, let $\mathfrak{M}_f Y$ be the path component in $\mathfrak{M}Y$ of f followed by the inclusion of Z in Y. Then $(\mathfrak{M}Y, \mathfrak{M}_f Y)$ is a pair of overspaces where $\mathfrak{M}q\colon \mathfrak{M}Y \to B$ is given by $(\mathfrak{M}q)(h) = h(x)$. Here $h \in \mathfrak{M}Y$, $x \in F$. Composition with an overmap $k\colon Y \to Y'$ defines an overmap $\mathfrak{M}k\colon \mathfrak{M}Y \to \mathfrak{M}Y'$. In this way $\mathfrak{M}$ is a functor.

Choose for R a coordinate cover $\{U_i\}_{i \in J}$ of B and coordinate transformations $g_{ij}\colon U_i \cap U_j \to G'$. Take the discrete topology on J, and let T be the subspace of $\mathfrak{M}_f Y \times J$ consisting of pairs (h, i) for which $(\mathfrak{M}q)(h) \in U_i$. Let $\mathfrak{M}_f^R Y$ be the space obtained from T by identifying (h, i), (h', j) when both the following conditions are met:

(1) $(\mathfrak{M}q)(h) = (\mathfrak{M}q)(h') = b'$ say.

(2) $h(x) = h'(g_{ji}(b') \cdot x)$ for all $x \in F$.

Then $\mathfrak{M}_f^R Y$ is an overspace of B, where $\mathfrak{M}^R q\colon \mathfrak{M}_f^R Y \to B$ is given by

$(\mathfrak{M}^R q)[h,i] = (\mathfrak{M}q)(h)$. Here $(h,i) \in T$. Composition with an overmap k: $Y \to Y'$ defines an overmap $\mathfrak{M}^R k$: $\mathfrak{M}_f^R Y \to \mathfrak{M}_{gf}^R Y'$ where g is the restriction of k to fibres. If Y is a Serre fibre space and if F is a CW-complex then, as in [4, I, §2], $\mathfrak{M}_f^R Y$ is a Serre fibre space. The following assertions are easy to verify.

(3) There is a natural homeomorphism *from* the space of extensions of f from fibres to overmaps $RF \to Y$ *to* the space of cross-sections s of $\mathfrak{M}_f^R Y$ for which $s(b) = f$.

(4) If $Y = RF'$ then $\mathfrak{M}_f^R Y$ is homeomorphic by an overmap with $R\mathcal{G}_f$.

3. **Localization.** We work in the category of pointed compactly generated spaces, and in the category of pointed overspaces of B. Let E, E' be Serre fibre spaces over B with *nilpotent* fibres [1, II.4.3] F, F'. Let $\{M, N\}$ be a partition of the primes.

By the *fibrewise localization* $\dot{E}_M$ *at* M of E we mean the same as in [1, I §8, V §4], except that we work with topological spaces in place of simplicial sets. We denote the localizing overmap $E \to \dot{E}_M$ by $\dot{\varepsilon}_M$, and its restriction $F \to F_M$ to fibres by ε_M. If an overmap f: $E \to E'$ restricts to h on fibres, we denote its fibrewise localization by $\dot{f}_M$, and we denote the localization of h by h_M. In this context it is customary to refer to 0 as a prime, and to talk of the fibrewise localization $\dot{E}_0$ of E at 0. I say that 0 is not a prime, and I talk instead of the fibrewise localization $\dot{E}_\varnothing$ of E at the empty set.

In the situation of §1 we prove the following result.

LEMMA 1. *If there is an overmap f: $P \to P$ of power $n \neq 1$, and if N contains only primes that divide n, then $\dot{P}_N$ has a cross-section.*

For this it suffices to prove that $\dot{P}_N | B^r$ has a cross-section for all $r \geqslant 1$. Here B^r is the r-skeleton of the CW-complex B. But G_N is path connected and so $\dot{P}_N | B^1$ has a cross-section. Suppose inductively that, for some $q \geqslant 1$, $\dot{P}_N | B^q$ has a cross-section s, and let $c(s) \in H^{q+1}(B; \pi_q \dot{G}_N)$ be the obstruction to extending $s | B^{q-1}$ to a cross-section of $\dot{P}_N | B^{q+1}$.

Consider the fibrewise localization $\dot{j}_N$: $\dot{P}_N \times_B (B \dot{\times} G)_N \to \dot{P}_N$ of the reverse G-action on P. Since

$$\dot{j}_N \times_B p_1 : \dot{P}_N \times_B (B \dot{\times} G)_N \to \dot{P}_N \times_B \dot{P}_N$$

is a weak equivalence over B, there is a cross-section s' of $(B \dot{\times} G)_N | B^q$ such that $(\dot{j}_N \times_B p_1)(s \times_B s')$ is homotopic through cross-sections to $s \times_B \dot{f}_N s$, by [3, 3.2]. Therefore $c(\dot{f}_N s) = c(s) + c(s')$.

By [8, 5.8.13] there is a weak equivalence $G_\varnothing \to K$ where K is a finite product of Eilenberg-Mac Lane groups $K(\mathbf{Q}, 2t + 1)$, and so $s' | B^{q-1}$ followed by $\dot{\varepsilon}_\varnothing$: $(B \dot{\times} G)_N \to (B \dot{\times} G)_\varnothing$ extends to a cross-section of $(B \dot{\times} G)_\varnothing | B^{q+1}$. Therefore $\varepsilon_{\varnothing**} c(s') = 0$ where $\varepsilon_{\varnothing**}: H^{q+1}(B; \pi_q G_N) \to H^{q+1}(B; \pi_q G_\varnothing)$ is induced by $\varepsilon_{\varnothing*}: \pi_q G_N \to \pi_q G_\varnothing$ on coefficients.

But $H^{q+1}(B; \pi_q G_N)$ is a finitely generated $\mathbf{Z}_N$-module and so $n^l c(s') = 0$ for some l. We have the following identities.

$$nc(\dot f_N^l s) = nc(\dot f_N^l s) = n^l(c(s) + c(s')),$$

$$n^l(c(s) + c(s')) = n^l c(s) = c(\dot f_N^l s).$$

These imply that $(n-1)c(\dot f_N^l s) = 0$ and, since multiplication by $n-1$ is an automorphism of $\pi_q G_N$, $c(\dot f_N^l s) = 0$. Hence $\dot f_N^l s | B^{q-1}$ extends to a cross-section of $\dot P_N | B^{q+1}$. This completes the induction, and the proof of Lemma 1.

4. **Proof of Theorem 2.** We regard $R = P \times_B P'$ as a principal $G \times G$-bundle, and we denote G by F, F' according to $G \times G$ acts by means of projection to the first, second factor. Let $s_m \colon G \to G$ be the mth power map, and denote $\mathcal{G}_{s_m}(F, F')$ by $\mathcal{G}_m$.

LEMMA 1. *If $\dot P_N$, $\dot P_{N'}$ have cross-sections then $(R\dot{\mathcal{G}}_m)_N$ has a cross-section for all m.*

To prove this note first that by means of the reverse G-actions on P, P' the given cross-sections produce weak equivalences

$$w\colon (B \dot\times G)_N \to \dot P_N, \quad w'\colon (B \dot\times G)_N \to \dot P'_N,$$

over B. By [3, 3.2] applied to the weak equivalence

$$\mathfrak{M}^R w\colon \mathfrak{M}^R_{\varepsilon_N}(B \dot\times G)_N \to \mathfrak{M}^R_{\varepsilon_N} \dot P_N,$$

and by (2.3), there is an overmap $f\colon P \to (B \dot\times G)_N$ such that wf is homotopic through overmaps to $\dot\varepsilon_N$. Corresponding under (2.3) to

$$P \xrightarrow{f} (B \dot\times G)_N \xrightarrow{(1 \dot\times s_m)_N} (B \dot\times G)_N$$

there is a cross-section of $\mathfrak{M}^R_{(s_m)_N \varepsilon_N}(B \dot\times G)_N$. Composing this cross-section with $\mathfrak{M}^R w'$, we get a cross-section s of $\mathfrak{M}^R_{(s_m)_N \varepsilon_N} \dot P'_N$.

The fiberwise localization at N of $\mathfrak{M}^{R\cdot}_{\varepsilon_N}\colon \mathfrak{M}^R_{s_m} P' \to \mathfrak{M}^R_{\varepsilon_N s_m} \dot P'_N$ is a weak equivalence. Therefore, by [3, 3.2] and since $\varepsilon_N s_m = (s_m)_N \varepsilon_N$, $(\mathfrak{M}^{R\cdot}_{s_m} P')_N$ has a cross-section. But by (2.4) $\mathfrak{M}^R_{s_m} P' = R\mathcal{G}_m$, and this completes the proof of Lemma 1.

LEMMA 2. *If there are overmaps $P \to P$, $P' \to P'$, $P' \to P$ of powers mn, mn, n, then there is an overmap $P \to P'$ of power m.*

To prove this note first that if $mn = 0$ then P, P' are trivial by (1.1). If $m = n = 1$ then Lemma 2 holds trivially, and if $m = n = -1$ then Lemma 2 is a consequence of [2, 6.3]. Suppose therefore that $mn \neq 0, 1$ and let N be the set of primes that divide n.

By (3.1) $\dot P_N$, $\dot P'_N$ have cross-sections, and so $(R\dot{\mathcal{G}}_m)_N$ has a cross-section by Lemma 1. By (2.3), (2.4), [6, 4.1], $(R\dot{\mathcal{G}}_m)_M$ has a cross-section. Hence, and by [6, 3.3], $R\mathcal{G}_m$ has a cross-section. Therefore, by (2.3), (2.4) again, there is an overmap $P \to P'$ of power m. This proves Lemma 2.

5. **Primes dividing** $\alpha(P)$. In the situation of §1 we prove the following result.

PROPOSITION 1. *The primes p that divide $\alpha(P)$ are precisely those for which the* *localization $\dot{P}_p$ does not have a cross-section.*

In order to prove Proposition 1 we require a lemma. For each positive integer i let $N(i)$ be a set of primes. Let $N = \bigcup_i N(i)$.

LEMMA 2. *If, for each i, $\dot{P}_{N(i)}$ has a cross-section then $\dot{P}_N$ has a cross-section.*

To prove Lemma 2 consider the commuting diagram

$$\begin{array}{ccc} \dot{P}_{N(i)} & \longrightarrow & (EG)_{N(i)} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\varepsilon_{N(i)}h} & (BG)_{N(i)} \end{array}$$

where $EG \to BG$ is a universal principal G-bundle and $h: B \to BG$ classifies P. Since $\dot{P}_{N(i)}$ has a cross-section and EG is contractible we have $\varepsilon_{N(i)}h \simeq \cdot$. Therefore by [1, V. 6.2] $\varepsilon_N h \simeq \cdot$, and so $\dot{P}_N$ has a cross-section. This proves Lemma 2.

Let N be the set of primes p such that $\dot{P}_p$ has a cross-section. Then $\dot{P}_N$ has a cross-section by Lemma 2. In the situation of §4 let P' be trivial. Then by (4.1) $(R\dot{\mathcal{G}}_m)_N$ has a cross-section for all m.

By obstruction theory either M is empty or there is a product m of primes in M such that $(P\dot{\mathcal{G}}_m)_M$ has a cross-section. If M is empty then P has a cross-section and $\alpha(P) = 1$. In the second case $R\mathcal{G}_m$ has a cross-section by [6, 3.3], and there is an overmap $P \to B \times G$ of power m by (2.3), (2.4). Let M', N' be the sets of primes that divide, do not divide $\alpha(P)$. Then we have shown that $M' \subseteq M$.

Now let $f: P \to B \times G$ be an overmap of power $\alpha(P)$. Since $\dot{f}_{N'}$ is a weak equivalence, $\dot{P}_{N'}$ has a cross-section. Hence $N' \subseteq N$. This completes the proof of Proposition 1.

As a consequence of Proposition 1 we have (1.3). Also $\dot{P}_\varnothing$ has a cross-section if and only if the rational cohomology characteristic classes of P are zero. Therefore Proposition 1 also implies (1.4).

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