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THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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ABSTRACT. The third order linear differential equation $Ly = y''' + p_2(t)y'' + p_1(t)y' + p_0(t)y = 0$ is studied, where $p_i(t)$ are continuous real-valued and periodic of period $\omega > 0$. Various criteria are obtained which guarantee "partial" asymptotic stability or instability by means of effective bounds on the Floquet characteristic multipliers of $Ly = 0$.

1. **Introduction**. Consider the third order differential equation

$$Ly = y''' + p_2(t)y'' + p_1(t)y' + p_0(t)y = 0 \tag{1.1}$$

where the $p_i(t)$ are continuous on $(-\infty, +\infty)$ and periodic of period $\omega > 0$. The stability properties of (1.1) are determined by the characteristic multipliers; i.e., by the roots of the Floquet characteristic equation

$$\lambda^3 - A_2\lambda^2 + A_1\lambda - B = 0 \tag{1.2}$$

where A_1, A_2 are given in terms of the fundamental system of solutions of (1) y_0, y_1, y_2, with $y_i^{(j)}(0) = \delta_{ij}$, $0 \leqslant i, j \leqslant 2$, as follows:

$$A_2 = y_0(\omega) + y_1'(\omega) + y_2''(\omega),$$

$$A_1 = \begin{vmatrix} y_0(\omega) & y_1(\omega) \\ y_0'(\omega) & y_1'(\omega) \end{vmatrix} + \begin{vmatrix} y_0(\omega) & y_2(\omega) \\ y_0''(\omega) & y_2''(\omega) \end{vmatrix} + \begin{vmatrix} y_1'(\omega) & y_0'(\omega) \\ y_1''(\omega) & y_2''(\omega) \end{vmatrix},$$

and $B = e^{-\omega J}$, $J = (1/\omega)\int_0^\omega p_2(t)\,dt$. In [12] stability of (1.1) is related to certain periodic and multi-point de la Valeé Poussin type boundary value problems and various necessary and/or sufficient conditions are obtained for stability and instability of (1.1). These problems were also considered in [2] via the method of lower and upper solutions for the corresponding second order Riccati equation

$$u'' + f(t, u, u') = 0, \qquad u = y'/y, \tag{1.3}$$

where

$$f(t, u, u') = 3uu' + p_2u' + u^3 + p_2u^2 + p_1u + p_0. \tag{1.4}$$

In this paper we shall be interested in obtaining lower and/or upper bounds on some or all of the characteristic multipliers which will be denoted

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by $\lambda_1, \lambda_2, \lambda_3$. Since $B = \lambda_1\lambda_2\lambda_3$ it is clear that a necessary condition for asymptotic stability (i.e., $|\lambda_i| < 1$, $i = 1, 2, 3$) is that $J > 0$. We shall be interested in finding sufficient conditions for "partial" asymptotic stability or instability. That is, we obtain conditions under which the asymptotically stable, stable, and unstable manifolds, denoted by $\mathcal{A}$, $\mathcal{S}$, and $\mathcal{U}$, respectively (i.e., the subspace of the solution space of (1.1) corresponding to characteristic multipliers of modulus $|\lambda| < 1$, $|\lambda| = 1$, $|\lambda| > 1$), have dimension k, $0 \leqslant k \leqslant 3$. We set $d_a = \dim \mathcal{A}$, $d_s = \dim \mathcal{S}$, and $d_u = \dim \mathcal{U}$. A solution of (1.1) will be said to be oscillatory if it has infinitely many zeros in a neighborhood of $+\infty$. The set of all oscillatory solutions will be denoted by $\mathcal{O}$. The set of all solutions of (1.1) which never vanish on $(-\infty, +\infty)$ will be denoted by $\mathcal{N}$.

If for each $0 \leqslant t < \omega$ the characteristic polynomial of (1.1),

$$\sigma(t, \rho) = \rho^3 + p_2(t)\rho^2 + p_1(t)\rho + p_0(t), \tag{1.5}$$

has real roots $\rho_i(t)$, $i = 1, 2, 3$, which are separated by constants (i.e., $\rho_1(t) \leqslant \mu_1 \leqslant \rho_2(t) \leqslant \mu_2 \leqslant \rho_3(t)$, $0 \leqslant t \leqslant \omega$), then (1.1) is disconjugate on $(-\infty, +\infty)$; that is, no nontrivial solution of (1.1) has more than two zeros on $(-\infty, +\infty)$. (See e.g., [1].) In this case, μ_1 and μ_2 are lower and upper solutions, respectively, of the Riccati equation (1.3) and the stability properties of (1.1) may be related to the sign of μ_1, μ_2 ([2]). We are interested here in obtaining criteria for the case when (1.1) has (perhaps) oscillatory solutions.

Consider the sets S_i, T_i, $i = 1, 2$, defined by

$$S_1 = \{\rho\colon \sigma(t, \mu) \leqslant 0 \text{ for all } \mu \leqslant \rho,\ t \in [0, \omega]\}, \tag{1.6}$$

$$S_2 = \{\rho\colon \sigma(t, \rho) \leqslant 0,\ t \in [0, \omega]\}, \tag{1.7}$$

$$T_1 = \{\rho\colon \sigma(t, \mu) \geqslant 0 \text{ for all } \mu \geqslant \rho,\ t \in [0, \omega]\}, \tag{1.8}$$

$$T_2 = \{\rho\colon \sigma(t, \mu) \geqslant 0,\ t \in [0, \omega]\}. \tag{1.9}$$

Let $\alpha_i = \sup S_i$, $\beta_i = \inf T_i$, $i = 1, 2$. Thus, $S_1 \subseteq S_2$, $T_1 \supseteq T_2$ and $\alpha_1 \leqslant \alpha_2$, $\beta_2 \leqslant \beta_1$ and $\sigma(t, \alpha_i) \leqslant 0 \leqslant \sigma(t, \beta_i)$, $i = 1, 2$. We shall see below that the stability properties of (1.1) can be related to the sets S_i, T_i, and that it is often possible to obtain effective lower bounds on α_1, α_2 and upper bounds on β_1, β_2 in terms of the coefficients $p_i(t)$. The main results are in §2 below and in §3 we briefly discuss their applicability.

2. We begin this section with a result which yields upper and lower bounds on the set of positive characteristic multipliers of (1.1). Since the coefficients in equation (1.2) are all real and $B = \lambda_1\lambda_2\lambda_3 > 0$, there always exists at least one positive characteristic multiplier.

THEOREM 2.1. *Let α_1, β_1 be defined as above (i.e., $\alpha_1 = \sup S_1$, $\beta_1 = \inf T_1$) and assume that for each fixed $\rho \in S_1 \cup T_1$, $\sigma(t, \rho) \not\equiv 0$, for $t \in [0, \omega]$. Assume further that for each $0 \leqslant \tau < \omega$ equation (1.1) is disconjugate on $[\tau, \tau + \omega]$. Then any positive characteristic multiplier λ of (1.1) satisfies*

$$e^{\alpha_1 \omega} < \lambda < e^{\beta_1 \omega}, \tag{2.1}$$

and the corresponding solution y_λ *of* (1.1) *belongs to* $\mathfrak{N}$.

PROOF. From Floquet theory, if $\lambda > 0$ is a multiplier of (1.1), then the corresponding solution of (1.1) may be written $y_\lambda(t) = \phi(t)e^{\rho t}$, where $\rho = \ln \lambda/\omega$, $\phi(t + \omega) = \phi(t)$ for all t, and $\phi(t) > 0$ on $[0, \omega]$ by disconjugacy. Thus, $y_\lambda \in \mathfrak{N}$. To prove (2.1) (i.e., $\alpha_1 < \rho < \beta_1$), suppose first that $\rho > \beta_1$. Then with $z(t) = e^{\rho t}$ we find that $Lz(t) = e^{\rho t}\sigma(t, \rho) \geqslant 0$. Let

$$k = \min_{0 \leqslant t \leqslant \omega} \phi(t) = \phi(t_0)$$

and let $v(t) = y_\lambda(t) - kz(t)$. Thus, $v(t) \geqslant 0$ on $[t_0, t_0 + \omega]$ and $v(t) = v'(t_0) = v(t_0 + \omega) = v'(t_0 + \omega) = 0$. Since $Ly = 0$ is disconjugate on $[t_0, t_0 + \omega]$, the Green's function $G_{21}(t, s)$ for the boundary value problem

$$Ly = f, \qquad y(t_0) = y'(t_0) = y(t_0 + \omega) = 0 \tag{2.2}$$

exists and is nonnegative for $t, s \in [t_0, t_0 + \omega]$. Furthermore, the solution of (2.2) may be written as

$$y(t) = \int_{t_0}^{t_0+\omega} G_{21}(t, s) f(s)\, ds. \tag{2.3}$$

Since $Lv(t) = -ke^{\rho t}\sigma(t, \rho) \equiv f(t) \leqslant 0$ on $[t_0, t_0 + \omega]$, we have

$$v(t) = \int_{t_0}^{t_0+\omega} G_{21}(t, s) f(s)\, ds \leqslant 0 \tag{2.4}$$

which implies $v(t) \equiv 0$, i.e., $\sigma(t, \rho) \equiv 0$, a contradiction to our assumption. Therefore, $\rho < \beta_1$. Similarly, with

$$v_1(t) = k_1 z(t) - y(t), \quad k_1 = \max_{0 \leqslant t \leqslant \omega} \phi(t) = \phi(t_1),$$

we may show that $\rho > \alpha_1$. This completes the proof.

As an immediate consequence of Theorem 2.1, we have

COROLLARY 2.2. *In addition to the hypotheses of Theorem* 2.1 *assume* $\beta_1 \leqslant 0$ ($\alpha_1 \geqslant 0$). *Then any positive characteristic multiplier* λ *of* (1.1) *satisfies* $\lambda < 1$, *and* $d_a \geqslant 1$, $\mathcal{Q} \cap \mathfrak{N} \neq \varnothing$ ($\lambda > 1$, $d_u \geqslant 1$ *and* $\mathfrak{U} \cap \mathfrak{N} \neq \varnothing$).

We now begin a more detailed analysis of $\mathfrak{U}$, $\mathfrak{S}$, and $\mathfrak{N}$. It is convenient to consider three cases: (i) $J > 0$, (ii) $J = 0$, (iii) $J < 0$. Since $B = e^{-\omega J} = \lambda_1\lambda_2\lambda_3$, it follows that $d_a = 3$ (i.e., asymptotic stability) occurs only in case (i), $d_s = 3$ (stability) occurs only in case (ii), and $d_u \geqslant 1$ (instability) always occurs in case (iii). Our primary concern is to obtain criteria under which d_a, d_s, d_u take on various values of k, $0 \leqslant k \leqslant 3$. In what follows, we will occasionally assume that the following condition holds (see also [12]):

CONDITION A. We say that equation (1.1) satisfies Condition A in case for any $0 \leqslant \tau < \omega$ the boundary value problem

$$Ly = 0, \qquad y(\rho) = y(\tau + \omega) = y(\tau + 2\omega) = 0$$

has a Green's function.

Clearly, Condition A will hold if, for example, equation (1.1) is disconjugate on $[\tau, \tau + 2\omega]$ for all $0 \leqslant \tau < \omega$. However, because of the particular properties of the three-point problem Condition A may also hold even though solutions of (1.1) exist with three zeros on $[\tau, \tau + 2\omega]$. Sufficient conditions for A to hold may be found in [12].

We begin with Case (i), $J > 0$:

THEOREM 2.3. *Let* $J > 0$ *and assume* (1.1) *is disconjugate on* $[\tau, \tau + \omega]$ *for all* $0 \leqslant \tau < \omega$.

(a) *If* $-J \leqslant \alpha_1 \leqslant \beta_1 \leqslant 0$ (*i.e.*, $(-\infty, -J] \subseteq S_1$, $[0, +\infty) \subseteq T_1$) *and if for each* $\rho \in (-\infty, -J] \cup [0, +\infty)$, $\sigma(t, \rho) \not\equiv 0$ *on* $[0, \omega]$, *then* $d_a \geqslant 2$, $d_s = 0$, *and* $\mathcal{Q} \cap \mathfrak{N} \neq \varnothing$. *Further, if Condition* A *holds, then* $d_a = 3$.

(b) *If* $[-J, 0] \subseteq S_2 \cup T_2$ *and* $\sigma(t, \rho) \not\equiv 0$ *on* $[0, \omega]$ *for each* $\rho \in [-J, 0]$ *then* $d_u \geqslant 1$, $d_a \geqslant 1$, $d_s = 0$, *and* $\mathfrak{N} \neq \varnothing$.

(c) *If Condition* A *holds, if* $(-\infty, 0] \subseteq S_1$ ($[-J, +\infty) \subseteq T_1$) *and* $\sigma(t, \rho) \not\equiv 0$ *on* $[0, \omega]$ *for* $\rho \leqslant 0$ ($\rho \geqslant -J$), *then* $d_a = 2$, $d_u = 1$ *and* $\mathcal{Q}$ *has a basis of solutions* u_1, u_2 *with* $u_i \in \mathcal{O}$ *whereas* $\mathfrak{U}$ *has a basis* $u_3 \in \mathfrak{N}$ ($d_a = 1$, $d_u = 2$, $\mathcal{Q} \cap \mathfrak{N} \neq \varnothing$ *and* $\mathfrak{U}$ *has a basis of oscillatory solutions*).

REMARK. The second part of Theorem 2.3(a) and part of Theorem 2.3(b) are essentially contained in results of [12].

PROOF. (a) Theorem 2.1 and Corollary 2.2 imply the existence of a multiplier λ_1 with $B < \lambda_1 < 1$ and the corresponding solution y_{λ_1} of (1.1) belongs to $\mathfrak{N}$. Moreover, all real positive multipliers satisfy $B < \lambda < 1$ by Corollary 2.2. Therefore, $\lambda_2\lambda_3 = B/\lambda_1 < 1$ so that one of λ_1 or λ_3 satisfies $|\lambda_i| < 1$. To be specific, assume $|\lambda_2| < 1$. Thus, $d_a \geqslant 2$ and $d_s = 0$ since complex multipliers occur only in conjugate pairs. This proves the first part of (a). Now if Condition A holds, then there are no negative multipliers. For if $\lambda < 0$ is a multiplier, then there exists a solution $y(t) \not\equiv 0$ of (1.1) with $y(t + \omega) = \lambda y(t)$ and hence $y(\tau) = 0$ for some $0 \leqslant \tau < \omega$. Thus, $y(\tau) = y(\tau + \omega) = y(\tau + 2\omega) = 0$, a contradiction. We conclude, therefore, that either all multipliers are real and satisfy $B < \lambda_i < 1$, $i = 1, 2, 3$, or λ_2, λ_3 are complex with $|\lambda_2| = |\lambda_3| < 1$, and in either case $d_a = 3$.

(b) If $[-J, 0] \subseteq S_2 \cup T_2$, then since $\sigma(t, \rho) \not\equiv 0$ for $\rho \in [-J, 0]$, it follows that $S_2 \cap T_2 = \varnothing$ so that $[-J, 0] \subseteq S_2$ or $[-J, 0] \subseteq T_2$. To be specific, assume $[-J, 0] \subseteq S_2$. If there exists a multiplier $\lambda \in [B, 1]$, then by Floquet theory $y_\lambda(t) = \phi(t)e^{\rho t}$, $\rho = \ln \lambda/\omega \in [-J, 0]$, and as in Theorem 2.1 $\phi(t) > 0$ on $[0, \omega]$. With $k = \max_{0 \leqslant t \leqslant \omega}\phi(t) = \phi(t_0)$ and $v(t) = ke^{\rho t} - y_\lambda(t) \geqslant 0$ on $[0, \omega]$ we have $Lv(t) = k\sigma(t, \rho) \leqslant 0$ and thus, as in Theorem 2.1, $v(t) \equiv 0 \Rightarrow \sigma(t, \rho) \equiv 0$, a contradiction. If $[-J, 0] \subseteq T_2$, then we use $v_1(t) = y_\lambda(t) - k_1e^{\rho t} \geqslant 0$, where $k_1 = \min_{0 \leqslant t \leqslant \omega}\phi(t) = \phi(t_1)$, and $L_3v_1 = -k_1\sigma(\rho, t) \leqslant 0 \Rightarrow v_1(t) \equiv 0$, a contradiction. It follows that no real multipliers λ satisfy $B \leqslant \lambda \leqslant 1$. Therefore, $0 < \lambda_1 < B$ or $\lambda_1 > 1$ so that $B/\lambda_1 = \lambda_2\lambda_3 > 1$ or < 1. Thus, $d_a \geqslant 1$, $d_u \geqslant 1$, and $d_s = 0$.

(c) If $(-\infty, 0] \subseteq S_1$, then this implies that there are no positive multipliers

$\leqslant 1$. Since there are no negative multipliers, $\lambda_1 > 1 > B$ so that $0 < \lambda_2\lambda_3 < 1$ and therefore λ_2, λ_3 must be complex with $|\lambda_2| = |\lambda_3| < 1$. Thus, the corresponding solutions are oscillatory. Similarly, if $[-J, +\infty) \subseteq T_1$, there are no positive multipliers $\geqslant B$ so that $0 < \lambda_1 < B$ and hence $\lambda_2\lambda_3 > 1$. Thus, λ_2, λ_3 are complex. This completes the proof.

A similar result may be obtained in the case $J = 0$.

THEOREM 2.4. *Let* $J = 0$ *and assume* (1.1) *is disconjugate on* $[\tau, \tau + \omega]$ *for all* $0 < \tau < \omega$.

(a) *If* $[0, +\infty) \subseteq T_1$, $\sigma(t, \rho) \not\equiv 0$ *for each* $\rho \in [0, +\infty)$, *and if Condition* A *holds, then* $d_u = 2$, $d_a = 1$, $\mathfrak{U}$ *has a basis of solutions* $u_1, u_2 \in \mathfrak{O}$ *and* $\mathfrak{A} \cap \mathfrak{N} \neq \varnothing$.

(b) *If* $(-\infty, 0] \subseteq S_1$, $\sigma(t, \rho) \not\equiv 0$ *for each* $\rho \in (-\infty, 0]$, *and if Condition* A *holds, then* $d_u = 1$, $d_a = 2$, $\mathfrak{A}$ *has a basis of solutions* $y_1, y_2 \in \mathfrak{O}$ *and* $\mathfrak{U} \cap \mathfrak{N} \neq \varnothing$.

(c) *If* $\sigma(t, 0) = p_0(t) \not\equiv 0$ *and* $0 \in S_2$ *or* $0 \in T_2$, *then* $d_u \geqslant 1$, $d_a \geqslant 1$, *and* $d_s = 0$.

PROOF. (a) If $[0, +\infty) \subseteq T_1$, Theorem 2.1 implies $\lambda_1 < 1 = B = \lambda_1\lambda_2\lambda_3$. Hence, $\lambda_2\lambda_3 \geqslant 1$. Condition A implies that there are no negative multipliers and since there are no real multipliers $\geqslant 1$, it follows that λ_2, λ_3 are complex with $|\lambda_2| = |\lambda_3| > 1$, and the corresponding solutions of (1.1) are oscillatory.

(b) If $(-\infty, 0] \subseteq S$, then by Theorem 2.1, $\lambda_1 \geqslant 1$ so that λ_2, λ_3 are complex with $|\lambda_2| = |\lambda_3| < 1$.

(c) The hypotheses imply, as in Theorem 2.3(b), that there are no real multipliers $\lambda = 1$. Hence, $\lambda_1 > 1 \Rightarrow \lambda_2\lambda_3 < 1 \Rightarrow |\lambda_2| < 1$ or $0 < \lambda_1 < 1 \Rightarrow |\lambda_2| > 1$. Therefore, $d_a \geqslant 1$, $d_u \geqslant 1$, and $d_s = 0$.

For the case $J < 0$ for which $d_u \geqslant 1$ always holds, we have the following result which gives criteria under which $d_u \geqslant 2$, $d_u = 3$, or $d_a \geqslant 1$. This result is analogous to Theorem 2.3.

THEOREM 2.5. *Let* $J < 0$ *and assume* (1.1) *is disconjugate on* $[\tau, \tau + \omega]$ *for all* $0 \leqslant \tau < \omega$.

(a) *If* $(-\infty, 0] \subseteq S_1$, $[-J, +\infty) \subseteq T_1$ (*i.e.*, $0 \leqslant \alpha_1 \leqslant \beta_1 \leqslant -J$) *and if for each* $\rho \in (-\infty, 0] \cup [-J, +\infty)$, $\sigma(t, \rho) \not\equiv 0$ *on* $[0, \omega]$, *then* $d_u \geqslant 2$, $d_s = 0$, *and* $\mathfrak{U} \cap \mathfrak{N} \neq \varnothing$. *Further, if Condition* A *holds, then* $d_u = 3$.

(b) *If* $[0, -J] \subseteq S_2 \cup T_2$ *and* $\sigma(t, \rho) \not\equiv 0$ *on* $[0, \omega]$ *for each* $\rho \in [0, -J]$, *then* $d_a \geqslant 1$, $d_u \geqslant 1$, $d_s = 0$ *and* $\mathfrak{N} \neq \varnothing$.

(c) *If Condition* A *holds, if* $(-\infty, -J] \subseteq S_1$ ($[0, +\infty) \subseteq T_1$) *and* $\sigma(t, \rho) \not\equiv 0$ *for* $\rho \leqslant -J$ ($\rho \geqslant 0$), *then* $d_u = 1$, $d_a = 2$, $\mathfrak{U} = \langle u_1 \rangle$, $u_1 \in \mathfrak{N}$, $\mathfrak{A} = \langle u_2, u_3 \rangle$, $u_2, u_3 \in \mathfrak{O}$ ($d_u = 2$, $d_a = 1$, $\mathfrak{U} = \langle u_1, u_2 \rangle$, $u_1, u_2 \in \mathfrak{O}$, $\mathfrak{N} = \langle u_3 \rangle$, $u_3 \in \mathfrak{N}$).

PROOF. (a) Theorem 2.1 implies $1 < \lambda_1 < B$ and $\mathfrak{U} \cap \mathfrak{N} \neq \varnothing$. Hence one of the remaining two multipliers, say λ_2, satisfies $|\lambda_2| > 1$. If Condition A holds, then there are no negative multipliers and therefore no real multipliers

$\leqslant 1$. This implies $|\lambda_i| > 1$, $i = 1, 2, 3$, so that $d_u = 3$.

(b) If $[0, -J] \subseteq S_2$ (or $\subseteq T_2$), then as in Theorem 2.3(b), no real multiplier satisfies $1 < \lambda < B$. Therefore, either $0 < \lambda_1 < 1$ or $\lambda_1 > B$ and in either case we have $d_a \geqslant 1$, $d_u \geqslant 1$, $d_s = 0$, and $\mathfrak{N} \neq \varnothing$.

(c) The proof is very similar to the proof of part (c) of Theorem 3.

In general, one cannot expect that $\beta_2 < \alpha_2$ holds, for the simple reason that this is sufficient to imply that $Ly = 0$ is disconjugate on $(-\infty, +\infty)$ and consequently all multipliers are real and positive. (Disconjugacy of $Ly = 0$ follows since β_2, α_2 are lower and upper solutions, respectively, for the Riccati equation (1.3); see [5], [3].) Thus, for example, $\alpha_1 \leqslant \beta_2 < \alpha_2 \leqslant \beta_1 \leqslant 0$ implies that $Ly = 0$ is disconjugate and hence there are no complex or negative multipliers; furthermore, no real multipliers are $\geqslant 1$ if $\sigma(t, \rho) \not\equiv 0$ for $\rho \geqslant 0$. This means $Ly = 0$ is asymptotically stable ($d_a = 3$) and, therefore, $J > 0$. Similarly $0 \leqslant \alpha_1 \leqslant \beta_2 < \alpha_2 \leqslant \beta_1$ and $\sigma(t, \rho) \not\equiv 0$ for $\rho \leqslant 0$ implies $d_u = 3$ and $J < 0$. Conditions which are sufficient to guarantee $d_s = 3$ in the case $J = 0$ are, in general, such that a small perturbation of the coefficients $p_i(t)$ leads to instability (i.e., the stability problem is not well-posed; see [10]). The next result yields conditions under which $d_a = d_u = d_s = 1$ in the case $J = 0$.

THEOREM 2.6. *Let* $J = 0$ *and assume* $p_2(t), p_0(t)$ *are odd and* $p_1(t)$ *is even. Assume further that* $\beta_2 < \alpha_2$. *Then* $0 < \lambda_2 < 1 = \lambda_1 < \lambda_3$ (*i.e.*, $d_a = d_s = d_u = 1$), *and* (1.1) *is disconjugate on* $(-\infty, +\infty)$.

PROOF. The remarks preceding the theorem imply disconjugacy of (1.1) on $(-\infty, +\infty)$ and hence that all multipliers are real and positive. It suffices, therefore, to show $\lambda_1 = 1$. But this is immediate since by the hypotheses, equation (1.1) is not changed by the transformation $t \to -t$ and therefore the coefficients A_1, A_2 in equation (1.2) satisfy $A_1 = A_2$. Since $B = 1$, it follows that $\lambda_1 = 1$ is a root of (1.2).

As our final result in this section, we have the following theorem which yields $d_a \geqslant 1$ with the assumption $p_2(t) \equiv 0$, by an appeal to a result on asymptotic behavior of nonoscillatory solutions for the case when oscillatory solutions exist. (See [6].)

THEOREM 2.7. *Assume* $p_0(t) > 0, p_1(t) \leqslant 0, p_2(t) \equiv 0$, *and that* (1.1) *is disconjugate on* $[\tau, \tau + \omega]$ *for all* $0 \leqslant \tau < \omega$. *Suppose also that* (1.1) *has a* (*nontrivial*) *solution* $u \in \mathcal{O}$. *Then*

(a) *any positive characteristic multiplier satisfies* $0 < \lambda < 1$; $d_a \geqslant 1$, $d_u \geqslant 1$ *and* (1.1) *has three linearly independent solutions* y_1, y_2, y_3 *with* $y_1 \in \mathcal{Q} \cap \mathfrak{N}$, $y_2, y_3 \in \mathcal{O}$;

(b) *if, in addition, Condition* A *holds, then* $d_a = 1$, $d_u = 2$ *and* $y_2, y_3 \in \mathfrak{U} \cap \mathcal{O}$.

PROOF. A result of Jones [6] implies that if $p_0 > 0, p_1 \leqslant 0$ and if (1.1) has oscillatory solutions, then any nontrivial nonoscillatory solution $y(t)$ satisfies $\lim_{t\to\infty} y(t) = 0$. Therefore, if $\lambda_1 > 0$ is a positive multiplier, then since the corresponding solution $y_1(t) = y_{\lambda_1}(t) = \phi(t)e^{\rho_1 t} \in \mathfrak{N}$, it follows that $\rho_1 =$

$\ln \lambda_1/\omega < 0$, $0 < \lambda_1 < 1$, so that $\lambda_2\lambda_3 > 1$ and hence either λ_2, λ_3 are both negative or complex. In any case, the corresponding solutions $y_2, y_3 \in \mathfrak{O}$. If Condition A holds, then λ_2, λ_3 must be complex so that $|\lambda_2| = |\lambda_3| > 1$. This completes the proof.

3. **Concluding remarks.** In this section we wish to briefly discuss the applicability of the results of §2.

If $p_0 \geq 0$ and $3p_1 \geq p_2^2$, then $\sigma_\rho(t, \rho) \geq 0$ for all ρ and hence $\sigma(t, \rho) \geq 0$ for all $\rho \geq 0$ so that $\beta_1 \leq 0$. If $p_0 \leq 0$ and $3p_1 \geq p_2^2$, then $\sigma(t, \rho) \leq 0$ for all $\rho \leq 0$ so that $\alpha_1 \geq 0$. Note also that $\sigma(t, \rho) \geq p_2\rho^2 + p_1\rho + p_0 \geq 0$ for all $\rho \geq 0$ provided $p_0 \geq 0, p_2 > 0$ and $4p_0p_2 \geq p_1^2$. Thus, $\beta_1 \leq 0$ in this case also. Similarly, $\sigma(t, \rho) \leq 0$ for all $\rho \leq 0$ if $p_0 \leq 0, p_2 < 0$ and $4p_0p_2 \geq p_1^2$ so that $\alpha_1 \geq 0$ in this case. Likewise, $0 \leq \alpha_1 \leq \beta_1 \leq 1$ provided $\sigma(t, 0) = p_0 \leq 0 \leq \sigma(t, 1) = 1 + p_2 + p_1 + p_0$ and $\sigma_\rho(t, \rho) \geq 0$ for $\rho \leq 0$ and $\rho \geq 1$. Analogously, $-1 \leq \alpha_1 \leq \beta_1 \leq 0$ if $\sigma(t, 0) = p_0 \geq 0 \geq \sigma(t, -1) = -1 + p_2 - p_1 + p_0$ and $\sigma_\rho(t, \rho) \geq 0$ for $\rho \leq -1$ and $\rho \geq 0$. Finally, we note that $\beta_2 \leq -1 < 1 \leq \alpha_2$ if $\sigma(-1, t) = -1 + p_2 - p_1 + p_0 \geq 0$ and $\sigma(1, t) = 1 + p_2 + p_1 + p_0 \leq 0$. This holds if $1 + p_1 \leq p_2 + p_0 \leq -(1 + p_1)$.

Sufficient conditions for Condition A to hold and for disconjugacy may be found in [12]. Other disconjugacy tests may be found in [1], [4], [7], [9], [11]. These tests combined with the remarks of the preceding paragraph allows one to give examples illustrating the results of §2. We leave this to the interested reader.

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