PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 64, Number 2, June 1977

ON BI-QUASITRIANGULAR OPERATORS[1]

L. J. GRAY

ABSTRACT. It is shown that the set of nilpotent operators T for which T^k has closed range for all k is norm dense in the set of all nilpotent operators. A consequence of this is that every bi-quasitriangular operator is a norm limit of operators which are similar to a direct sum of weighted shifts plus scalars.

1. **Introduction.** For $\mathcal{H}$ a complex separable Hilbert space, denote by $B(\mathcal{H})$ the collection of bounded linear operators on $\mathcal{H}$. Denote by $\mathcal{A}$, $\mathfrak{N}$, and BQT the subsets of $B(\mathcal{H})$ consisting of the algebraic, nilpotent, and bi-quasitriangular operators, respectively. Voiculescu [5] has shown that $\overline{\mathcal{A}}$ = BQT, where the bar denotes norm closure. In this note, we show that one can get by with a slightly smaller set than $\mathcal{A}$, namely $\overline{\mathcal{A}_c} = \overline{\mathcal{A}}$, where $\mathcal{A}_c$ is the set of algebraic operators T such that $(T-\lambda)^j$ has closed range for all complex λ and positive integers j. The operators in $\mathcal{A}_c$ are of a particularly simple form: they are similar to Jordan operators. Thus, if $T \in \mathcal{A}_c$, there exists an invertible X such that $T = XSX^{-1}$, $S = \sum_{j=1}^{\infty} \oplus S_j$, and each S_j operates on C^{n_j}, $0 < n_j < \infty$, by the matrix

$$\begin{bmatrix} \lambda_j & 1 & 0 & 0 & \cdots & 0 \\ & \lambda_j & 1 & 0 & \cdots & . \\ & & & & & \\ & & & & & 1 \\ & 0 & & & & \\ & & & & & \lambda_j \end{bmatrix},$$

where λ_j is an eigenvalue of T (see [2], [3], or [6]).

To show that $\mathcal{A}_c$ is norm dense in $\mathcal{A}$, it clearly suffices to show that $\mathfrak{N}_c = \mathfrak{N} \cap \mathcal{A}_c$ is norm dense in $\mathfrak{N}$. This result may be of interest for problems concerning nilpotent approximations (see [1]).

2. For $Y \in B(\mathcal{H})$, R_Y and N_Y will denote the orthogonal projection onto the closure of the range of Y and the null space of Y, respectively. If Q is a projection, Q will also denote the subspace R_Q; it will be clear from the

Received by the editors September 30, 1976.
AMS (MOS) subject classifications (1970). Primary 47B99.
Key words and phrases. Bi-quasitriangular, nilpotent operator, algebraic operator, closed range.

[1]By acceptance of this article, the publisher or recipient acknowledges the U. S. Government's right to retain a nonexclusive, royalty-free license in and to any copyright covering the article.

context which is meant. If P and Q are two subspaces, $\theta(P,Q)$ will be the angle between them; i.e.

$$\theta(P,Q) = \inf\left(\cos^{-1}|(p,q)|\right)$$

where $p \in P$, $q \in Q$ and $\|p\| = \|q\| = 1$. Finally, for $T \in \mathfrak{N}$, $i(T)$ will be the smallest integer n such that $T^n = 0$.

LEMMA 1. *Suppose that* $A \in B(\mathcal{H})$ *and that* Q *is a projection. Then for any* $\varepsilon > 0$ *there exists an operator* A_0 *such that*:

(a) $\|A - A_0\| < \varepsilon$,

(b) A_0 *has closed range and* $N_{A_0} = N_A$,

(c) $\theta(Q, R_{A_0} - R_{A_0} \cap Q) > 0$,

(d) $R_{(1-Q)A} = R_{(1-Q)A_0}$.

PROOF. We first note that, by using the polar decomposition, it is easy to see that any operator Y can be uniformly approximated by operators X with closed range such that $R_Y = R_X$ and $N_Y = N_X$. Thus, we may assume that A has closed range. Let δ be the greatest lower bound of A (i.e. $\|Ax\| \geqslant \delta\|x\|$, $x \in N_A^{\perp}$) and set $\varepsilon_0 = \min(\varepsilon/2, \delta/4)$. Now, apply the above remark to the operators QA and $(1 - Q)A$ to obtain A_1 and A_2 with closed range such that $\|QA - A_1\| < \varepsilon_0$, $\|(1 - Q)A - A_2\| < \varepsilon_0$ and $R_{A_1} = R_{QA}$, $R_{A_2} = R_{(1-Q)A}$ and similarly for the null spaces. If we now set $A_0 = A_1 + A_2$, (a) and (d) are obviously satisfied.

To prove (b) note that $N_{A_0} = N_{A_1} \cap N_{A_2} = N_{QA} \cap N_{(1-Q)A} = N_A$ and, thus, it remains to verify that A_0 is bounded below on $N_A^{\perp}$. If $x \in N_A^{\perp}$, $\|x\| = 1$, then $\|Ax\| \geqslant \delta$, and we may assume $\|QAx\| \geqslant \delta/2$ (if not, then $\|(1 - Q)Ax\| \geqslant \delta/2$). Since $\|QA - A_1\| < \delta/4$, we have $\|A_1x\| \geqslant \delta/4$, and thus $\|A_0x\| \geqslant \delta/4$. This verifies (b), and now (c) follows from the fact that $(1 - Q)A_0 = A_2$ has closed range.

LEMMA 2. *If* $T \in \mathfrak{N}$ *and* $\varepsilon > 0$, *there exists* $T_0 \in \mathfrak{N}$ *such that*

(a) $\|T - T_0\| < \varepsilon$, $i(T) = i(T_0)$,

(b) T_0 *has closed range*,

(c) $R_T = R_{T_0}$.

PROOF. The proof is by induction on $i(T)$; assume that the lemma is true for $i(T) \leqslant n - 1$. Let $i(T) = n$ and write $\mathcal{H} = R_T \oplus N_{T^*}$. With respect to this decomposition, T has the matrix representation

$$\begin{pmatrix} T' & A \\ 0 & 0 \end{pmatrix}$$

where $T' = T|_{R_T}$. Thus, T' is nilpotent and $i(T') = n - 1$; let $T_0' \in \mathfrak{N}$ be such that $\|T' - T_0'\| < \varepsilon/2$ and properties (a)–(c) are satisfied. We let A_0 be the operator guaranteed by Lemma 1, with $Q = R_{T_0'}$, such that $\|A - A_0\| < \varepsilon/2$. We now set

$$T_0 = \begin{pmatrix} T_0' & A_0 \\ 0 & 0 \end{pmatrix}$$

(we note that since

$$R_{T_0'} < R_T \quad \text{and} \quad R_{(1-R_{T_0'})A} = R_{(1-R_{T'})A_0},$$

the range of A_0 is contained in R_T). We have $T_0 \in \mathfrak{N}$ because $T_0' \in \mathfrak{N}$, and clearly, (a) is satisfied. Note that $R_{T_0} \subset R_T$ is a consequence of the way we constructed T_0, while the reverse inclusion follows from the fact that $R_{T_0'} = R_{T'}$ and property (d) of Lemma 1. Thus, (c) is verified and it remains to show that T_0 has closed range. But

$$R_{T_0'} + R_{A_0} \subset \text{range of } T_0 \subset R_{T_0} = \sup(R_{T_0'}, R_{A_0})$$

and thus we must show that $R_{T_0'} + R_{A_0}$ is closed. However, this follows from $\theta(R_{T_0'}, R_{A_0} - R_{A_0} \cap R_{T_0'}) > 0$ and problem 3, p. 243 of [4].

THEOREM 1. *$\mathfrak{N}_c$ is norm dense in $\mathfrak{N}$.*

PROOF. We will show that for $T \in \mathfrak{N}$ and an integer k, there exists $T_0 \in \mathfrak{N}$ with $i(T_0) = i(T)$, $\|T - T_0\| < \varepsilon$ and such that for $1 \leqslant j \leqslant k$, T_0^j has closed range and $R_{T_0^j} = R_{T^j}$. Letting $k = i(T)$ will complete the proof.

The above statement for $k = 1$ is the content of Lemma 2; assume it is true for $k \leqslant n - 1$. We proceed as in Lemma 2: write

$$T = \begin{pmatrix} T' & A \\ 0 & 0 \end{pmatrix}.$$

To construct T_0, replace T' by the T_0' assured by the induction hypothesis, and replace A by the A_0 of Lemma 1, again with $Q = R_{T_0'}$. That T_0 has closed range equal to R_T follows as in Lemma 2. Furthermore,

$$\text{range}(T_0^2) = \text{range}(T_0'|_{R_{T_0}}) = \text{range}(T_0'|_{R_T}) = R_{T_0'}$$

and, thus, in succession, we obtain

$$\text{range}(T_0^j) = \text{range}(T_0|_{R_{T_0^{j-1}}}) = \text{range}(T_0|_{R_{(T_0')^{j-2}}}) = R_{(T_0')^{j-1}}$$

for $i \leqslant j \leqslant n$. Thus, T^j has closed range.

COROLLARY 1. *$\mathfrak{A}_c$ is norm dense in $\mathfrak{A}$.*

COROLLARY 2. *Every operator in $\overline{\mathfrak{N}}$ is the norm limit of operators similar to nilpotent weighted shifts.*

COROLLARY 3. *Every bi-quasitriangular operator is a norm limit of operators which are similar to an operator of the form $\sum_{k=1}^{n} \oplus W_k$, where W_k is of the form weighted shift + $\lambda_k I$.*

REFERENCES

1. C. Apostol and N. Salinas, *Nilpotent approximations and quasinilpotent operators*, Pacific J. Math. **61** (1975), 327–337.

2. C. Apostol and J. G. Stampfli, *On derivation ranges*, Indiana J. Math. **25** (1976), 857–870.

3. L. J. Gray, *Jordan representation for a class of nilpotent operators*, Indiana J. Math. **26** (1977), 57–64.

4. A. E. Taylor, *Functional analysis*, Wiley, New York, 1967.

5. D. Voiculescu, *Norm-limits of algebraic operators*, Rev. Roumaine Math. Pures Appl. **19** (1974), 371–378. MR **49** #7826.

6. L. Williams, *Similarity invariants for a class of nilpotent operators* (to appear).

COMPUTER SCIENCES DIVISION, UNION CARBIDE CORPORATION, NUCLEAR DIVISION,* OAK RIDGE, TENNESSEE 37830

*Prime contractor for the U. S. Energy Research and Development Administration.