

ON WEITZENBÖCK'S THEOREM IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let k be an algebraically closed field and let $f: G_a \rightarrow \text{GL}(V)$ be a finite-dimensional k -rational representation of the additive group G_a . If the subspace of G_a -fixed points in V is a hyperplane, then the ring of G_a -invariant polynomial functions on V is finitely generated over k . This result is an analog of a classical theorem of Weitzenböck, a modern proof of which has been given by C. S. Seshadri.

Introduction. Let k be an algebraically closed field and G_a the one-dimensional vector group over k . Let G_a act linearly on the finite-dimensional k -vector space V so that the subspace of fixed points on V has codimension one. The purpose of this note is to show that, in this case, the ring of G_a -invariant polynomial functions on V is finitely generated over k . This result is an analog of a classical result of Weitzenböck, a modern proof of which is due to C. S. Seshadri [6].

If a rational representation of G_a on V factors through an $\text{SL}(2, k)$ representation, then the representation is called *fundamental*. Seshadri [6] gives a proof of Weitzenböck's theorem for fundamental representations and shows that every representation of G_a in characteristic zero is fundamental. An example is given here which shows that this is not the case in positive characteristics. In particular, there exist representations of G_a having fixed point loci of codimension one which are not fundamental.

Notations and conventions. Throughout, k denotes a fixed algebraically closed field of arbitrary characteristic. All algebraic groups are affine k -groups, all varieties and morphisms are defined over k , and representations of algebraic groups are assumed k -rational. A point of a variety is always a k -rational point.

Let $\rho: G_a \rightarrow \text{GL}(V)$ be a given representation; i.e., a homomorphism of algebraic groups. Give V the structure of an affine space with ring of functions $S(V^*)$ —the symmetric algebra on the k -dual of V . The set V_0 of G_a -fixed points of V is a linear subspace of V . Suppose that V_0 has codimension one in V . Let $\{e_1, \dots, e_n\}$ be a basis of V such that $\langle e_1, \dots, e_{n-1} \rangle = V_0$. Let $\{x_1, \dots, x_n\}$ be the corresponding dual basis. Then $S(V^*) \cong$

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$k[x_1, \dots, x_n]$ and, since G_a must act trivially on V/V_0 , the canonical action of G_a on $k[x_1, \dots, x_n]$ is as follows:

$$(1) \quad \begin{aligned} t \cdot x_i &= x_i + a_i(t)x_n, & 1 < i < n - 1, \\ t \cdot x_n &= x_n, & \text{all } t \in G_a. \end{aligned}$$

Here $a_i(T) \in k[G_a] = k[T]$ is an additive function on G_a , so it is a p -polynomial in $k[T]$ if $p = \text{char } k > 0$. The following lemma is well known, but we give a proof for lack of a precise reference.

LEMMA 1. *Let $A \subseteq B$ be integral domains with A normal and B integral over A . Let E be a set of ring endomorphisms of B . Denote by A^E and B^E the subring of E -invariant elements of A and B , respectively. Then B^E is integral over A^E .*

PROOF. Let $b \in B^E$ and let $\underline{P}(T) = T^m + a_1T^{m-1} + \dots + a_m$ be the minimal monic polynomial of b over the quotient field of A . Since A is normal, all the a_i belong to A . Now, if $e \in E$, then

$$0 = e(P(b)) = b^m + e(a_2)b^{m-1} + \dots + e(a_m)$$

and hence, $a_i = e(a_1), \dots, a_m = e(a_m)$.

Suppose $\text{char } k = p > 0$. Let D be the noncommutative ring of p -polynomials over k (i.e., composition of functions as the law of multiplication). It is well known (cf. [4] or [5, Theorem 1]) that D is a right (and left) Euclidean domain. Consequently, every nonzero left ideal is principal and of finite codimension as a k -module. It follows that if a and b are nonzero elements of D , then $Da \cap Db \neq \{0\}$ and, hence, a and b have a nonzero left least common multiple.

We are now prepared to prove our main result.

THEOREM. *Let V be a finite-dimensional rational G_a -module and suppose the set of fixed points in V has codimension one in V . Then the ring of G_a -invariant polynomial functions on V is finitely generated over k .*

PROOF. We need only consider the case $\text{char } k = p > 0$. We may assume $k[V] = k[x_1, \dots, x_n]$ where the coordinate functions x_1, \dots, x_n satisfy (1). Note that, by a simple change of basis of V_0 , we may assume that the degrees of all the $a_i(T)$ in (1) are equal. By the preceding remarks, we know that for each i there exist a pair of nonzero p -polynomials (b_i, d_i) such that $b_i \circ a_1 = d_i \circ a_i$. For $2 < i < n - 1$ set

$$(2) \quad z_i = d_i(x_i/x_n) - b_i(x_1/x_n).$$

It is easy to verify that $z_i \in k(x_1, \dots, x_n)^{G_a}$. Since the degree of a_1 equals the degree of a_i , the degrees of b_i and d_i must be the same for any given i . If this degree is p^n , then multiplying (2) by $x_n^{p^n}$ gives the invariant polynomial

$$(3) \quad y_i = x_i^{p^r} + \sum_{j=0}^{r_i-1} r_{ij}(x_1, x_n)x_i^{p^j}, \quad 2 \leq i \leq n-1,$$

where $r_{ij}(x_1, x_n) \in k[x_1, x_n]$.

It follows from (3) that $k[x_1, \dots, x_n]$ is integral over

$$S = k[y_2, \dots, y_{n-1}, x_1, x_n].$$

Moreover, S is a polynomial ring over k , hence, S is normal. Now $k[x_1, \dots, x_n]^{G_a}$ is integrally closed and by Lemma 1, integral over S^{G_a} . But $S = k[y_1, \dots, y_{n-1}, x_n][x_1]$ and, since y_2, \dots, y_{n-1} and x_n are invariants and k is infinite, it follows that

$$S^{G_a} = k[y_1, \dots, y_{n-1}, x_n].$$

By [7, p. 267, Theorem 9], $k[x_1, \dots, x_n]^{G_a}$ is finitely generated over k . Q.E.D.

We show now that the theorem does in fact give an extension of Seshadri's result. Assume that $p = \text{char } k > 2$. Let x, y and z be coordinates on $V_0 = k^3$ and let G_a act on V_0 via the assignments:

$$t \cdot x = x + t^p z, \quad t \cdot y = y + tz, \quad t \cdot z = z \quad \text{all } t \in G_a.$$

Note that $z = 0$ defines the fixed point locus on V_0 . Moreover, if $ax + by + cz = 0$ is a G_a -stable hyperplane, then $at^p + bt + c = c$, all $t \in G_a$. Hence, $a = b = 0$ and so $z = 0$ is the unique G_a -stable hyperplane in V_0 .

We claim that V_0 is not a fundamental G_a -module. We need the following lemma.

LEMMA 2. *Suppose $\text{char } k = p > 2$. Then every 3-dimensional rational $\text{SL}(2, k)$ -module is completely reducible.*

PROOF. Let V be a 3-dimensional rational $\text{SL}(2, k)$ -module. If V is simple or trivial there is nothing to prove. Since V is completely reducible if and only if V^* is, we may assume that V contains a 2-dimensional nontrivial submodule W . Then W is necessarily irreducible and the action of $\text{SL}(2, k)$ on W is given by an i th iterate of the Frobenius map composed with the identity representation (cf. [1]). Let $\sigma = -\text{Id} \in \text{SL}(2, k)$. Then σ is represented by the matrix

$$M_\sigma = \begin{bmatrix} -1 & 0 & \alpha \\ 0 & -1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to some basis of V . Since the minimal polynomial for M_σ is $T^2 - 1$, M_σ is diagonalizable, so it is represented by

$$M' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in a suitable basis of V . But σ lies in the centre of $\text{SL}(2, k)$ and the centralizer of M' in $\text{GL}(3, k)$ is

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & * \end{bmatrix}, A \in \mathrm{GL}(2, k) \right\}.$$

Thus $V \hat{=} W \oplus k$ and the lemma is proved.

Now if V is a 3-dimensional fundamental G_a -module, then either V is irreducible as an $\mathrm{SL}(2, k)$ -module, or $V \cong W \oplus k$ as an $\mathrm{SL}(2, k)$ -module, where W is a nontrivial $\mathrm{SL}(2, k)$ -module of dimension 2. In the first case, we see, by [1], that V is isomorphic to an iterate of the Frobenius composed with the irreducible representation of $\mathrm{SL}(2, k)$ on the space of forms of degree two in two variables. For this module the action of G_a is given by the matrix representation

$$t \rightarrow \begin{bmatrix} 1 & 2t^{p'} & t^{2p'} \\ 0 & 1 & t^{p'} \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that the fixed point locus is a line, not a plane. In the second case, the action of G_a is given by a matrix representation of the form

$$t \rightarrow \begin{bmatrix} 1 & t^{p'} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The fixed point locus for this action is, indeed, a hyperplane. However, there are infinitely many G_a -stable hyperplanes. It follows that, in any case, V_0 is not a fundamental G_a -module.

REMARKS. 1. Suppose that $\mathrm{char} k = 0$ and G_a acts linearly on V so that V^{G_a} has codimension one in V . Then, using arguments entirely analogous to those given above, one can show that $k[V]^{G_a}$ is a polynomial algebra over k . This seems unlikely in positive characteristic, but at present we know of no counterexamples. In general, $k[V]^{G_a}$ is not a polynomial algebra [2].

2. It is known (cf. [2, Remark 7]) that, for fundamental G_a -actions, $k[V]^{G_a}$ is the coordinate ring of a rational variety; i.e. the quotient field of $k[V]^{G_a}$ is purely transcendental over k . Moreover, if $\mathrm{char} k = 0$, it follows from Seshadri's proof of Weitzenböck's theorem that $k[V]^{G_a}$ is actually the ring of invariants of an $\mathrm{SL}(2, k)$ action on a larger polynomial algebra; hence, the recent results of Hochster and Roberts [3] imply that, in this case, $k[V]^{G_a}$ is a Cohen-Macaulay ring. It would be interesting to know if either or both of these results hold in positive characteristic for the 'codimension one' actions discussed in this paper.

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REFERENCES

1. A. Borel, *Linear representations of semi-simple algebraic groups*, Proc. Sympos. Pure Math., vol. 29, Amer. Math. Soc., Providence, R. I., 1975, pp. 421-440.

2. A. Fauntleroy, *Linear G_a -actions on affine spaces and associated rings of invariants*, J. Pure Appl. Algebra **9** (1977).
3. M. Hochster and J. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. **13** (1974), 115–175.
4. T. Kambayashi, M. Miyanishi and M. Takeuchi, *Unipotent algebraic groups*, Lecture Notes in Math., vol. 414, Springer-Verlag, Berlin and New York, 1974.
5. O. Ore, *On a special class of polynomials*, Trans. Amer. Math. Soc. **35** (1933), 559–584.
6. C. S. Seshadri, *On a theorem of Weitzenböck*, J. Math. Kyoto **1–3** (1962).
7. Samuel Zariski, *Commutative algebra*, Vol. I, Van Nostrand, Princeton, N. J., 1960.

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