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A CARDINAL INEQUALITY FOR TOPOLOGICAL SPACES INVOLVING CLOSED DISCRETE SETS

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ABSTRACT. Let X be a T_1 topological space. Let $a(X) = \sup\{\alpha: X$ has a closed discrete subspace of cardinality $\alpha\}$ and $v(X) = \min\{\alpha: \Delta_X$ can be written as the intersection of α open subsets of $X \times X\}$; here Δ_X denotes the diagonal $\{(x, x): x \in X\}$ of X. It is proved that $|X| \leqslant \exp(a(X)\, v(X))$. If, in addition, X is Hausdorff, then X has no more than $\exp(a(X)\, v(X))$ compact subsets.

1. **Introduction.** There are several known relationships among the cardinal functions on a topological space that involve the cardinalities of closed discrete subsets of the space. Among these are Jones's lemma (see, for example, [10, p. 100]) and the recent results of Burke and Hodel [2]. It is the purpose of this note to add to this list of relationships.

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Information about cardinal functions on topological spaces appears in Juhasz [6]; we shall use the notation and terminology of this text. We shall henceforth assume that all hypothesized topological spaces are T_1. Any additional separation axioms used in the proof of a theorem will be set forth explicitly in the statement of the theorem. The cardinality of a set X is denoted by $|X|$; $[X]^2$ will denote the set of two-element subsets of X. Cardinal numbers are identified with the set of ordinals preceding them. The smallest cardinal greater than the cardinal λ is denoted by λ^+. We shall use the following set-theoretic theorem, due to Erdös and Rado; see [4].

1.1. THEOREM. *Let α be a cardinal number and let X be a set such that $|X| > \exp \alpha$. Let $(A_i)_{i<\alpha}$ be a collection of no more than α subsets of $[X]^2$ such that $[X]^2 = \bigcup \{A_i: i < \alpha\}$. Then there exists an $i_0 < \alpha$ and a subset S of X such that $[S]^2 \subset A_{i_0}$ and $|S| = \alpha^+$.*

2. **The main results.** Let $a(X)$ and $v(X)$ be as defined in the abstract.

2.1. THEOREM. $|X| \leqslant \exp(a(X)v(X))$.

PROOF. Since X is T_1, Δ_X can be written as the intersection of some family

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of open subsets of $X \times X$, so $v(X)$ is well defined. Suppose $\Delta_X = \bigcap\{G_i: i < v(X)\}$ where each G_i is open in $X \times X$. For each $i < v(X)$ and $x \in X$ there exists an open subset $V_i(x)$ of X such that $(x,x) \in V_i(x) \times V_i(x) \subset G_i$. Thus without loss of generality we can assume that for each $i < v(X)$, $G_i = \bigcup\{V_i(x) \times V_i(x): x \in X\}$.

Now suppose the theorem is false, i.e. that $|X| > \exp(a(X)v(X))$. For each $i < v(X)$ define A_i to be $\{\{x,y\} \in [X]^2: (x,y) \notin G_i\}$. Since $\bigcap\{G_i: i < v(X)\} = \Delta_X$ it follows that $[X]^2 = \bigcup\{A_i: i < v(X)\}$. By Theorem 1.1 there exists $i_0 < v(X)$ and $S \subset X$ such that $|S| = (v(X)a(X))^+$ and $[S]^2 \subset A_{i_0}$. We will show that S is a closed discrete subset of X; since $|S| > a(X)$, this gives a contradiction and the theorem will be proved.

Let $z \in X$ and suppose z were a limit point of S. As X is T_1 each neighborhood of z meets infinitely many members of S. In particular, there exist distinct points x and y in $S \cap V_{i_0}(z)$. Thus $\{x,y\} \in [S]^2 - A_{i_0}$, which is a contradiction. Thus S has no limit points in X; equivalently, S is a closed discrete subset of X. □

Note that neither $a(X)$ nor $v(X)$ can be omitted from the exponent. If D is a large discrete space then $|D| > \exp(v(D))$, while a large countably compact space Y satisfies $|Y| > \exp(a(Y))$. Furthermore it is not possible to replace $v(X)$ by the character $\chi(X)$. To see this first note that if X is countably compact and first countable then $\exp(\chi(X)a(X)) = \exp(\aleph_0)$. Then observe that the subspace of $(\exp \aleph_0)^+$ (equipped with the order topology) consisting of all ordinals of countable cofinality is countably compact, first countable, and has cardinality $(\exp \aleph_0)^+$.

There are two immediate corollaries to 2.1.

2.2. COROLLARY. *If X is Lindelöf and has a G_δ diagonal then* $|X| \leqslant \exp(\aleph_0)$.

2.3. COROLLARY. *If X is collectionwise Hausdorff then* $|X| \leqslant \exp(v(X)c(X))$ (*where $c(X)$ is the cellularity of X, i.e.* $c(X) = \sup\{|\mathcal{G}|: \mathcal{G}$ *is a family of pairwise disjoint nonempty open subsets of* $X\}$).

PROOF. In a collectionwise Hausdorff space $a(X) \leqslant c(X)$. □

Corollary 2.2 is of interest in relation with a recent problem concerning the cardinality of Lindelöf spaces. By Arhangel'skiĭ's theorem [1], Lindelöf first countable spaces have cardinality at most $\exp(\aleph_0)$. The question of whether Lindelöf spaces whose points are $G_\delta s$ have cardinality at most $\exp \aleph_0$ remains open. Corollary 2.2 gives another class of Lindelöf spaces whose points are $G_\delta s$ (in addition to first countable spaces and hereditarily Lindelöf spaces) for which the inequality is valid–those with G_δ diagonals.

2.4. EXAMPLE. In 2.3 we have shown that the cardinal inequality $|X| \leqslant \exp(v(X)c(X))$ holds for a rather broad class of spaces. However, it does not hold for all Hausdorff spaces; in particular, the Katětov extension $\mathcal{K}\underline{N}$ of the countable discrete space $\underline{N}$ does not satisfy this inequality (see [7], [6, p. 64] for a discussion of the Katětov extension). Recall that the underlying set of $\mathcal{K}\underline{N}$ is the set $\beta\underline{N}$ of all ultrafilters of $\underline{N}$ (with principal ultrafilters being identified with points of $\underline{N}$), topologized by decreeing that each point of $\underline{N}$ is isolated and $\{\{p\} \cup A: \overline{A} \in p\}$ is a neighborhood base for p if $p \in \beta\underline{N} - \underline{N}$. As $\mathcal{K}\underline{N}$ is separable, $c(\mathcal{K}\underline{N}) = \aleph_0$. For each $p \in \beta\underline{N} - \underline{N}$ fix $A(p) \in p$ and let $A(p) - \{0, \ldots, n\}$ be denoted by $A(p,n)$. Then

$$\Delta_{\mathcal{K}\underline{N}} = \cap\left\{\cup\{\Delta_N \cup A(p,n) \times A(p,n): p \in \beta\underline{N} - N\}: n \in N\right\},$$

so $v(\mathcal{K}\underline{N}) = \aleph_0$. However, as is well known (see, for example, 9.2 of [5]), $|\mathcal{K}\underline{N}| = \exp(\exp \aleph_0)$. Obviously $\mathcal{K}\underline{N}$ is Hausdorff.

It is also worth noting that $\mathcal{K}\underline{N}$ provides a counterexample to the question, posed by Simon [8, p. 209] of whether the weight of X is always no greater than $v(X)c(X)$.

2.5. *Question.* Although 2.4 provides an example of a Hausdorff space X for which $|X| \leqslant \exp(v(X)c(X))$ is untrue, we know of no regular Hausdorff space X for which the inequality fails. It would be interesting to know if the inequality is always true for regular Hausdorff spaces.

We conclude this note by obtaining a bound on the number of compact subsets of a Hausdorff space. Let $\mathcal{K}(X)$ denote the set of compact subsets of X.

2.6. Theorem. *Let X be Hausdorff. Then* $|\mathcal{K}(X)| \leqslant \exp(a(X)v(X))$.

Proof. First note that if K is a compact Hausdorff space then $w(K) \leqslant v(K)$. ($w(K)$ denotes the weight of K.) This can be proved by generalizing in the obvious fashion the proof (suggested in [3, problem C, p. 183]) of Šneĭder's theorem that a compact Hausdorff space with a G_δ diagonal is second countable (and therefore metrizable); see [9].

Now let $K \in \mathcal{K}(X)$. Then $v(K) \leqslant v(X) \leqslant \alpha$, so as noted above, $w(K) \leqslant \alpha$. Thus $d(K) \leqslant \alpha$ ($d(K)$ denotes the density character of K). Thus there is a one-to-one mapping from $\mathcal{K}(X)$ into the set of all subsets of X of cardinality no greater than α. Since by 2.1, $|X| \leqslant 2^\alpha$, there are no more than $(2^\alpha)^\alpha = 2^\alpha$ such subsets of X. The theorem follows. □

2.7. Remark. Other recent results concerning the cardinality of $\mathcal{K}(X)$ appear in [2].

References

1. A. V. Arhangel'skiĭ, *The power of bicompacta with first axiom of countability*, Dokl. Akad. Nauk. SSSR **187** (1969), 967–968 = Soviet Math. Dokl. **10** (1969), 951–955.

2. D. K. Burke and R. E. Hodel, *The number of compact subsets of a topological space*, Proc. Amer. Math. Soc. **58** (1976), 363–368.

3. R. Engelking, *Outline of general topology*, North-Holland, Amsterdam, 1968.

4. P. Erdös and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489.

5. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960.

6. I. Juhasz, *Cardinal functions in topology*, Math. Centre Tract, no. 34, Math. Centrum, Amsterdam, 1971.

7. M. Katětov, *Über H-abgeschlossene und bikompakte Räume*, Časopis Pěst. Mat. **69** (1940), 36–49.

8. P. Simon, *A note on cardinal invariants of square*, Comment Math. Univ. Carolinae **14** (1973), 205–213.

9. V. E. Šneĭder, *Continuous images of Suslin and Borel sets. Metrization theorems*, Dokl. Akad. Nauk. SSSR **50** (1945), 77–79. (Russian)

10. S. Willard, *General topology*, Addison-Wesley, Reading, Mass., 1968.

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