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ON NONOSCILLATORY LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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ABSTRACT. For the equation $y'' + q(t)y = 0$, we derive necessary conditions for nonoscillation in terms of "general means" of $Q(t) = \int_0^t q\,ds$.

1. **Main theorem.** The object of this note is to obtain necessary conditions for

$$y'' + q(t)y = 0, \tag{1.1}$$

$q \in C^0[0, \omega)$, to be nonoscillatory at $t = \omega$ ($\leq \infty$). The result generalizes [2] (cf. [3, §7, pp. 362–369]) involving the case $\omega = \infty$ and the arithmetic mean

$$t^{-1}\int_0^t Q(s)\,ds, \quad \text{where } Q(t) = \int_0^t q(s)\,ds.$$

A generalization of [2] was given by Coles and Willett in which $\omega = \infty$, and the arithmetical mean was replaced by a more general mean

$$(KQ)(t) = \int_0^t K(t, s)Q(s)\,ds, \tag{1.2}$$

where $K(t, s)$ is the kernel $K_n(t, s) = K_n(t, s; \phi_1, \ldots, \phi_n)$,

$$K_n(t, s) = \phi_1(s)\int_{s<s_2<\cdots<s_n<t}\cdots\int \phi_2(s_2)\cdots\phi_n(s_n)\,ds_2\cdots ds_n\Big/k_n(t),$$
$$\int_0^t K_n(t, s)\,ds = 1, \tag{1.3}$$

depending on nonnegative functions $\phi_1, \ldots, \phi_n$; cf. [1], [5] and references there, and see §4 below. The essential arguments in these papers were the same as in [2], but they also involved many formal calculations with iterated integrals which seemed to have nothing to do with (1.1). In this paper, we show that the methods of [2] can be used for very general "means" (1.2) and, in fact, the object in part is to state suitable conditions on the kernel $K(t, s) \geq 0$ for which the arguments of [2] are applicable. Roughly, the conditions imposed on K will include the following three types: first, a generalization of the condition that the map $Q \mapsto KQ$ in (1.2) is regular in the

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sense of (Toeplitz) summation theory [cf. (a), (b*) below]; second, that this map is a bounded integral operator from the Banach space $(L^{2/\sigma} \cap C^0)[0, \omega)$ to the Banach space $(L_0^\infty \cap C^0)[0, \omega)$ for some σ, $1 \leqslant \sigma \leqslant 2$, [cf. ($c_\sigma$) below] and, third, a quite technical condition given in (d) below. Our main result will be stated as a sufficient condition for (A_τ):

(A_τ) Let τ be fixed, $1 \leqslant \tau \leqslant 2$. Let $K(t, s) \geqslant 0$ be continuous for $0 \leqslant s \leqslant t$, $0 < t < \omega$ ($\leqslant \infty$). We say that the kernel K has property (A_τ) if the following holds: If $q(t) \in C^0[0, \omega)$, then a necessary condition for (1.1) to be nonoscillatory at $t = \omega$ is either that

$$\liminf_{t \to \omega} \int_0^t K(t, s) Q(s)\, ds = -\infty, \tag{1.4}$$

or that there exists a constant C such that

$$\int_0^t K(t, s) |C - Q(s)|^\tau ds \to 0 \quad \text{as } t \to \omega. \tag{1.5}$$

Furthermore, when (1.1) is nonoscillatory, then (1.5) holds if and only if one and/or every solution $y \not\equiv 0$ of (1.1) satisfies

$$\int^\omega (y'/y)^2\, dt < \infty. \tag{1.6}$$

REMARK. We do not gain generality by replacing (1.1) by an equation of the form $(P(t)y')' + R(t)y = 0$, $P(t) > 0$, since the latter can be reduced to (1.1) by the change of independent variables $t \to u$, $du = dt/P(t)$, and $K(t, s)$ replaced by $K(t, s)P(s)$.

We use the letter K for both the kernel $K(t, s)$ and the corresponding integral operator $Q \mapsto KQ$ in (1.2). In what follows, we always assume that $K(t, s) \geqslant 0$ is continuous for $0 \leqslant s \leqslant t$, $0 < t < \omega$. We list some other conditions on K which will be assumed from time to time:

(a) $$(KQ)(t) \to 0, \quad t \to \omega, \quad \text{if } Q \in (L_0^\infty \cap C^0)[0, \omega),$$

i.e., the operator K is a bounded map on the Banach space $(L_0^\infty \cap C^0)[0, \omega)$.

(b) $$(KQ)(t) \to \infty, \quad t \to \omega, \quad \text{if } C^0[0, \omega) \ni Q(t) \to \infty,\ t \to \omega.$$

(c) $$(KQ)(t) \to 0, \quad t \to \omega, \quad \text{if } Q \in (L^{2/\sigma} \cap C^0)[0, \omega)$$

for some fixed σ, $1 \leqslant \sigma \leqslant 2$; i.e., the map K is a bounded operator from the Banach space $(L^{2/\sigma} \cap C^0)[0, \omega)$ to the Banach space $(L_0^\infty \cap C^0)[0, \omega)$ for some σ, $1 \leqslant \sigma \leqslant 2$.

(d) Assume that there exist functions $0 < k(t) \in C^0[0, \omega)$, $0 \leqslant m(t) \in C^0[0, \omega)$, a kernel $K_0(t, s) \geqslant 0$ continuous for $0 \leqslant s \leqslant t$, $0 < t < \omega$, and a constant α, $0 \leqslant \alpha \leqslant 2$, with the following properties:

$$\kappa(t, s) = \frac{\partial}{\partial t} \int_s^t k(t) K(t, s)\, ds \geqslant 0 \tag{1.7}$$

exists and is continuous, $0 \leqslant s \leqslant t$, $0 < t < \omega$;

(1.8) $K^{2-\alpha}(t, s)/K_0(t, s)$ is continuous for $0 \leqslant s \leqslant t$, $0 < t < \omega$;

(1.9) $\int^{\omega} m(t)\,dt = \infty$ or $(\infty >) \limsup_{t\to\omega} k(t)\int_t^{\omega} m(t)\,dt > 0,$

and, finally, (1.7) satisfies for s near ω, $s \leq t < \omega$,

(1.10) $\kappa(t,s) \geq k^2(t)m(t)\left(\int_0^t K^{2-\alpha}(t,u)/K_0(t,u)\,du\right)K^{\alpha}(t,s)K_0(t,s).$

REMARKS. We give examples and illustrations of these conditions in §4, but we make some simple comments here. Condition (a) holds if and only if K satisfies the two conditions

(a_1) $$\int_0^t K(t,s)\,ds = O(1) \quad \text{as } t \to \omega,$$

(a_2) $$\int_0^T K(t,s)\,ds = o(1), \quad t \to \omega, \quad \text{for fixed } T,\ 0 < T < \omega,$$

while (a_2) is implied by

(a_2^*) $K(t,s)\,ds = o(1), \quad t \to \omega,$ uniformly for $0 \leq s \leq T$.

When (a), i.e., (a_1)–(a_2), holds, condition (b) is satisfied if

(b^*) $$\int_0^t K(t,s)\,ds \geq c > 0 \quad \text{for } t \text{ near } \omega,\ c = \text{const.}$$

Also, when (a_2) holds, then condition (c_σ) is equivalent to

(1.11) $K(t,s) = O(1), \quad s \to \omega,$ uniformly in $t \geq s$, $\sigma = 2$,

(1.12) $$\int_s^t |K(t,u)|^{2/(2-\sigma)}\,du = O(1), \quad s \to \omega,$$ uniformly in $t \geq s$, $1 \leq \sigma < 2$.

If $k \in C^1$, then (1.9) holds, when either

(1.9*) $\int^{\omega} m(t)\,dt = \infty$ or $k(t) \to \infty$ and $k'/k^2 = O(m)$ as $t \to \omega$.

Also, if $K_0(t,s) = \kappa(t,s)$ satisfies (1.8) with $\alpha = 0$, then (1.10) holds with

(1.10*) $$m(t) = 1/k^2(t)\int_0^t \left[K^2(t,u)/\kappa(t,u)\right]du.$$

THEOREM. *Let* $q \in C^0[0,\omega)$ *be such that* (1.1) *is nonoscillatory at* $t = \omega$. (i) *If* K *satisfies* (a) *and* (c_σ), *and* (1.6) *holds for some solution* $y \not\equiv 0$ *of* (1.1), *then* (1.5) *holds for* $1 \leq \tau \leq \sigma$, *with*

(1.13) $$C = y'(a)/y(a) + Q(a) - \int_a^{\omega} (y'/y)^2\,ds$$

if $y(t) \neq 0$ *for* $(0 \leq)a \leq t < \omega$; *in particular,*

(1.14) $$\liminf_{t\to\omega} \int_0^t K(t,s)Q(s)\,ds > -\infty.$$

(ii) *If* K *satisfies* (a), (b), (d) *and if* (1.14) *holds, then every solution* $y \not\equiv 0$ *of* (1.1) *satisfies* (1.6).

If (b*) *also holds in* (i), *then of course*

$$C = \lim_{t\to\omega} \int_0^t K(t,s)Q(s)\,ds \Big/ \int_0^t K(t,u)\,du. \tag{1.15}$$

The proof of the theorem follows [2]. It obviously implies

COROLLARY. *Conditions* (a), (b), (c_σ), (d) *imply* (A_σ).

2. **Proof of** (i). Let $y \not\equiv 0$ be a solution of (1.1) satisfying (1.6). We shall verify (1.5) for $\tau = \sigma$. The cases $1 \leqslant \tau < \sigma$ then follow from Hölder's inequality and (a_1), while (1.14) is implied by the case $\tau = 1$ of (1.5) and (a_1). Also (1.13) will be clear from the proof.

Suppose that $y \neq 0$ for $t \geqslant a$ ($\geqslant 0$), so that $r = y'/y$ satisfies the Riccati equation

$$r' + r^2 + q(t) = 0 \tag{2.1}$$

for $t \geqslant a$. A quadrature gives

$$r(s) + \int_a^s r^2(u)\,du = r(a) - Q(s) + Q(a). \tag{2.2}$$

In view of (1.6), we can write this as

$$r(s) - \int_s^\omega r^2(u)\,du = C - Q(s), \tag{2.3}$$

where C is given by (1.13). Consequently, by (2.3),

$$|C - Q(s)|^\sigma \leqslant 2^\sigma r^\sigma(s) + 2^\sigma\left(\int_s^\omega r^2(u)\,du\right)^\sigma,$$

so that $K \geqslant 0$ gives

$$\int_a^t K(t,s)|C - Q(s)|^\sigma\,ds$$
$$\leqslant 2^\sigma \int_a^t K(t,s)r^\sigma(s)\,ds + 2^\sigma \int_a^t K(t,s)\left(\int_s^\omega r^2(u)\,du\right)^\sigma ds.$$

By (a), the last term on the right tends to 0 as $t \to \omega$. Also, by (c_σ), the first term tends to 0 as $t \to \omega$. This proves (i).

3. **Proof of** (ii). Suppose (ii) is false, so that (1.6) fails for some solution $y \not\equiv 0$ of (1.1). Let $t = a$ be fixed, sufficiently near to ω, so that, in particular, $y(t) \neq 0$ for $t \geqslant a$. Using (2.2),

$$\int_a^t K(t,s)r(s)\,ds + S(t) = c(t,a) - \int_0^t K(t,s)Q(s)\,ds, \qquad t \geqslant a, \tag{3.1}$$

where we put

$$S(t) = \int_a^t K(t,s)\left(\int_a^s r^2(u)\,du\right) ds, \tag{3.2}$$

$$c(t,a) = [r(a) + Q(a)]\int_a^t K(t,s)\,ds + \int_0^a K(t,s)Q(s)\,ds. \tag{3.3}$$

Since (1.6) does not hold, (b) implies that

$$S(t) \to \infty \quad \text{as } t \to \omega. \tag{3.4}$$

Note that $c(t, a)$ is bounded as $t \to \omega$ by assumption (a_1). Hence (1.14) implies that the right side of (3.1) is bounded from above as $t \to \omega$, so that

$$-\int_a^t K(t, s) r(s)\, ds > S(t)/2 \quad \text{for } t \text{ near } \omega. \tag{3.5}$$

For a fixed constant α, Schwarz's inequality gives

$$\begin{aligned}&\left|\int_a^t K(t, s) r(s)\, ds\right|^2 \\ &\quad < \left(\int_a^t \left[K^{2-\alpha}(t, u)/K_0(t, u)\right] du\right)\left(\int_a^t K^\alpha(t, s) K_0(t, s) r^2(s)\, ds\right),\end{aligned} \tag{3.6}$$

whenever the right side is meaningful. The last two inequalities show that

$$\begin{aligned}&4k^2(t)\left(\int_a^t \left[K^{2-\alpha}(t, u)/K_0(t, u)\right] du\right)\left(\int_a^t K^\alpha(t, s) K_0(t, s) r^2(s)\, ds\right) \\ &\quad > k^2(t) S^2(t).\end{aligned} \tag{3.7}$$

By Fubini's theorem, (3.2) can be rewritten

$$S(t) = \int_a^t \left[\int_s^t K(t, u)\, du\right] r^2(s)\, ds. \tag{3.8}$$

Hence, by (1.7),

$$\left[k(t) S(t)\right]' = \int_a^t \kappa(t, s) r^2(s)\, ds, \tag{3.9}$$

so that (1.10) and (3.7) imply

$$4\left[k(t) S(t)\right]' > m(t)\left[k(t) S(t)\right]^2 \quad \text{for } t \text{ near } \omega$$

(if $t = a$ has been fixed sufficiently near ω). By a quadrature, for t near ω,

$$4/S(t) > k(t) \int_t^u m(s)\, ds \quad \text{if } t < u < \omega.$$

This contradicts the first alternative in (1.9) if $u \to \omega$; and, by (3.4), contradicts the second alternative if $u \to \omega$ and then $t \to \omega$. This proves (ii).

4. **Remarks and examples**. The kernel $K(t, s) = 1/t$ for $0 < s < t$, $t > 0$, $\omega = \infty$ of [2] satisfies (a), (b), (c_2), (d) with $k(t) = t$, $m(t) = 1/t$, $K_0 \equiv 1$, and $\alpha = 1$.

EXAMPLE 1. More generally, let $0 < \phi(t) \in C^0[0, \omega)$,

$$K(t, s) = \phi(t - s)/k(t), \quad \text{where } k(t) = \int_0^t \phi(s)\, ds. \tag{4.1}$$

Thus (a_1) and (b^*) are satisfied, and (a_2) is equivalent to

$$\int_{t-T}^t \phi(s)\, ds \Big/ \int_0^t \phi(u)\, du \to 0 \quad \text{as } t \to \omega,\ 0 < T < \omega. \tag{4.2}$$

Note that

$$\kappa(t, s) = \frac{\partial}{\partial t}\int_s^t \phi(t-u)\,du = \frac{\partial}{\partial t}\int_0^{t-s}\phi(u)\,du = \phi(t-s).$$

Hence, if $\alpha = 1$ and $K_0 \equiv 1$, then (1.10) holds with

$$m(t) = 1/k(t) = 1 \Big/ \int_0^t \phi(u)\,du, \tag{4.3}$$

and so, (1.9) holds if either

$$\int^\omega \left[dt \Big/ \int_0^t \phi(u)\,du\right] = \infty \quad \text{or}$$
$$\limsup_{t\to\omega}\left(\int_0^t \phi(s)\,ds\right)\int_t^\omega \left[du \Big/ \int_0^u \phi(r)\,dr\right] > 0. \tag{4.4}$$

This holds if either

$$\int^\omega \left[dt \Big/ \int_0^t \phi(u)\,du\right] = \infty \quad \text{or}$$
$$\int^\omega \phi\,ds = \infty \quad \text{and} \quad \phi(t) = O\left(\int_0^t \phi\,du\right) \quad \text{as } t\to\omega. \tag{4.5}$$

Also, (1.11), hence (c_2), holds if

$$\phi(u) = O\left(\int_0^{u+s}\phi\,dv\right), \quad s\to\omega, \quad \text{uniformly for } 0 \leqslant u < \omega, \tag{4.6}$$

and (1.12), hence (c_σ), $1 \leqslant \sigma < 2$, holds if

$$\int_0^u \phi^{2/(2-\sigma)}\,dv = O\left(\int_0^{u+s}\phi\,dv\right)^{2/(2-\sigma)}, \quad s\to\omega, \tag{4.7}$$
$$\text{uniformly for } 0 \leqslant u < \omega.$$

EXAMPLE 2. Let $n \geqslant 1$ and, if $n = 1$, replace all $(n-1)$-fold integrals below by 1. Let $0 < \phi_1, \ldots, \phi_n \in C^0[0, \omega)$, and consider the kernel in (1.3), i.e.,

$$K(t, s) = L(t, s)/k(t), \tag{4.8}$$

where

$$L(t, s) = \phi_1(s)\int_{s<s_2<\cdots<s_n<t}\cdots\int \phi_2(s_2)\cdots\phi_n(s_n)\,ds_2\cdots ds_n, \tag{4.9}$$

(4.10)

$$k(t) = \int_0^t L(t, s)\,ds = \int_{0<s_1<\cdots<s_n<t}\cdots\int \phi_1(s_1)\cdots\phi_n(s_n)\,ds_1\cdots ds_n.$$

Then (a_1), (b^*) hold, and (a_2^*) is equivalent to

$$L(t, s) = o(k(t)) \quad \text{as } t\to\omega, \quad \text{uniformly for } 0 \leqslant s \leqslant T, \tag{4.11}$$

$0 < T < \omega$ fixed. By repeated applications of L'Hôpital's rule, (4.11) holds if

$$k_m(t) \to \infty \quad \text{as } t\to\omega \text{ for } m = 1, \ldots, n, \tag{4.12}$$

where $k = k_n$ in (4.10). Note that

(4.13)

$$\kappa(t,s) = \phi_n(t)\int_{s<s_1<\cdots<s_{n-1}<t}\cdots\int \phi_1(s_1)\cdots\phi_{n-1}(s_{n-1})\,ds_1\cdots ds_{n-1}.$$

If $\alpha = 1$ and $K_0 = 1$, then (1.10) in (d) holds for $T \leq s \leq t < \omega$ if

$$m(t) = \Big\{\sup_{T\leq s\leq t}\left[\kappa(t,s)/L(t,s)\right]\Big\}/k(t). \tag{4.14}$$

If $\alpha = 0$ and $K_0 = \kappa$, then (1.10) holds if

$$m(t) = 1\Big/\int_0^t\left[L^2(t,u)/\kappa(t,u)\right]du, \tag{4.15}$$

by the remark concerning (1.10*). Also, (1.11), hence (c_2), holds if (4.11) and

$$L(t,s) = O(k(t)), \quad s\to\omega, \quad \text{uniformly in } t \geq s, \tag{4.16}$$

and (1.12), hence (c_σ), $1 \leq \sigma < 2$, holds if (4.11) and

$$\int_s^t\left[L(t,u)\right]^{2/(2-\sigma)}du = O\left(k^{2/(2-\sigma)}(t)\right), \quad s\to\omega, \quad \text{uniformly in } t \geq s. \tag{4.17}$$

Consider the case $n = 1$ and let $\phi = \phi_1$. If $\phi(t) \leq c_0$, then (4.14) can be replaced by

$$m(t) = \phi(t)\Big/c_0\int_0^t\phi(s)\,ds,$$

so that the first alternative in (1.9) holds by (4.12_n). Without any assumption of boundedness, (4.15) is

$$m(t) = \phi(t)\Big/\int_0^t\phi^2(s)\,ds,$$

and (1.9) becomes

$$\int^\omega\left[\phi(t)\Big/\int_0^t\phi^2\,du\right]dt = \infty \quad \text{or}$$

(4.18)

$$\limsup_{t\to\omega}\left(\int_0^t\phi\,ds\right)\int_t^\omega\left[\phi(u)\Big/\int_0^u\phi^2\,dv\right]du > 0,$$

which is satisfied if (4.12_n) and

$$\int^\omega\left[\phi(t)\Big/\int^t\phi^2\,du\right]dt = \infty \quad \text{or} \quad \int_0^t\phi^2\,ds = O\left(\int_0^t\phi\,du\right)^2 \quad \text{as } t\to\omega.$$

Finally, (4.16), (4.17) reduce, respectively, to

$$\phi(t) = O\left(\int_0^t\phi\,du\right) \quad \text{as } t\to\omega,$$

$$\int_s^t\phi^{2/(2-\sigma)}\,du = O\left(\int_0^t\phi\,du\right)^{2/(2-\sigma)}, \quad s\to\omega, \quad \text{uniformly in } t \geq s.$$

The above conditions seem to be weaker than those imposed by Coles and Willett [1]. For example, when $n = 2$, (4.15) becomes

$$m(t) = \phi_2(t) \bigg/ \int_0^t \left[\phi_1^2(s) \left(\int_s^t \phi_2 \, du \right)^2 \bigg/ \left(\int_s^t \phi_1 \, du \right) \right] ds,$$

which is required to satisfy (1.9). In Coles and Willett [1], the analogous function is

$$\tilde{m}(t) = \phi_2(t) \bigg/ \int_0^t \phi_1^2(s) \left[\int_s^t \phi_2^2(u) \, du / \phi_1(u) \right] ds.$$

By Schwarz's inequality, $m(t) \geqslant \tilde{m}(t)$, so that (1.9) is weaker than the corresponding condition in [1]. Furthermore (4.17), with $\sigma = 1$, is a weaker condition than the corresponding one in [1].

EXAMPLE 3 (ADDED 2/14/77). Using the procedure of [2], Macki and Wong [4] consider necessary conditions for (1.1) to be nonoscillatory [in fact, for analogues of (A_τ)] in which (1.6) is replaced by

$$\int^{\omega} P(z'/z)^2 \, dt, \quad \text{where } z = y/P^{1/2}, \tag{4.19}$$

$\omega = \infty$, $0 < P \in C^1[0, \omega)$, Q in (1.2) and (1.4)–(1.5) is replaced by

$$Q(t) = \int_0^t Pq \, ds - \int_0^t (P'/2P^{1/2})^2 \, ds + P'(t)/2, \tag{4.20}$$

and K is a kernel of the type (1.3) in Example 2 with $n = 1$. Their results are included in the theorem above after suitable changes of variables; cf. the remark following (1.6). Suppose for a moment that $P \in C^2$ and make the change of dependent variables $y = zP^{1/2}$, so that (1.1) becomes $(Pz')' + Rz = 0$, or

$$d^2z/du^2 + Rz = 0, \quad \text{where } R = qP + P^{1/2}(P^{1/2})'', \; 0 \leqslant u < \omega^*, \tag{4.21}$$

$u = u(t)$ is a new independent variable, $du = dt/P$, $u(0) = 0$, $\omega^* = u(\omega) \leqslant \infty$. Actually, the existence of P'' is not needed, for the proof does not involve the equation (4.21) nor the corresponding Riccati equation, but the integrated form of the latter [cf. (2.2)],

$$r(u) + \int_a^u r^2(\sigma) \, d\sigma = c - Q(t(u)), \tag{4.22}$$

where Q is given by (4.20), $t(u)$ is the inverse of $u = u(t)$, and $r(u) = z^{-1}(dz/du) = Pz'/z$ at $t = t(u)$; cf. [4] for another derivation of (4.22). Thus the analogue of the theorem holds if the kernel $K(t(v), t(u))P(t(u))$ satisfies the corresponding conditions with $0 \leqslant s \leqslant t$, $0 < t < \omega$ replaced by $0 \leqslant u \leqslant v$, $0 < v < \omega^*$, and (1.6) by (4.19).

Macki and Wong also formulated conditions analogous to (a) and (c), with $\sigma = 2$, for their kernel.

References

1. W. J. Coles and D. Willett, *Summability criteria for oscillation of second order linear differential equations*, Ann. Mat. Pura Appl. **79** (1968), 391–398.

2. P. Hartman, *On non-oscillatory linear differential equations of second order*, Amer. J. Math. **74** (1952), 389–400.

3. ______, *Ordinary differential equations*, S. M. Hartman, Baltimore, 1973.

4. J. W. Macki and J. S. W. Wong, *Oscillation theorems for linear second order ordinary differential equations*, Proc. Amer. Math. Soc. **20** (1969), 67–72.

5. D. Willett, *On the oscillatory behavior of the solutions of second order linear differential equations*, Ann. Polon. Math. **21** (1969), 175–194.

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