PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 64, Number 2, June 1977

ON THE CHROMATIC NUMBER OF SUBGRAPHS OF A GIVEN GRAPH

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ABSTRACT. It is shown that for arbitrary positive integers m,n there exists a $\phi(m,n)$ such that if $\chi(G) > \phi(m,n)$, then G contains either a complete subgraph of size m or a subgraph H with $\chi(H) = n$ containing no C_3. This gives an answer to a problem of Erdös and Hajnal.

Introduction. In [1], [2], [3] P. Erdös raised the following question: Is it true that to every n there is an $f(n)$ such that if $\chi(G) > f(n)$ then G contains a subgraph H in the sense defined below such that $\chi(H) = n$ and H contains no C_3 (the circuit of size 3) ? In this paper we prove this conjecture.

We prove the following

THEOREM. *For arbitrary positive integers m,n there exists a $\phi(m,n)$ such that if $\chi(G) > \phi(m,n)$ then G contains either a complete subgraph K_m of size m or a subgraph H with $\chi(H) > n$ containing no C_3.*

The conjecture of Erdös and Hajnal now follows easily. Tutte proved [5] that there exists a graph H with $\chi(H)$ arbitrarily large, which contains no C_3. Now given a graph G_1 with $\chi(G) \geqslant \phi(m,n)$, then either G contains K_m and we find a triangle-free subgraph of K_m with big chromatic number, or G contains H as above. We make no attempt here to find the smallest $\phi(m,n)$ with the desired property.

Notation. Given a graph $G = (V,E)$ we shall sometimes write $V = V(G)$, $E = E(G)$. H is a subgraph of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We shall denote this fact by $H \subset G$. Let G be a graph and let the set $V(G)$ be linearly ordered by $<$. By a left neighborhood of a vertex v we shall understand the graph $L(v,G)$ where

$$V(L(v,G)) = \{v';v' < v, \{v',v\} \in E(G)\},$$
$$E(L(v,G)) = E(G)/V(L(v,G)).$$

PROOF OF THE THEOREM. In the proof we use the following proposition of Zykov [4]. Let G be a graph with $\chi(G) \geqslant (p-1)(q-1)+1$. Let $E(G) = E_1 \cup E_2$ be a partition of $E(G)$. Then either $\chi((V(G),E_1)) \geqslant p$ or $\chi((V(G),E_2)) > q$. (This follows since $p-1$ coloring and a$(q-1)$ coloring

Received by the editors April 6, 1976.
AMS (MOS) subject classifications (1970). Primary 05C15.
Key words and phrases. Graph, subgraph, triangle-free graph, chromatic number.

of the blocks together yield a $(p-1)(q-1)$ coloring of the whole by pairs of colors.)

Obviously we can choose $\phi(2,n) = 2$ for every n.

Let us suppose that there exists $\phi(m-1,n)$ i.e., every graph G with $\chi(G) \geqslant \phi(m-1,n)$ contains either K_{m-1} or triangle-free H with $\chi(H) \geqslant n$. Let us consider a graph G_0 with

$$\chi(G_0) = (n-1)^{\phi(m-1,n)-1} + 1 = \phi(m,n)$$

and with $K_m \not\subset G_0$. Let $<$ be a linear order of $V(G_0)$. Since $K_m \not\subset G_0$ we have $K_{m-1} \not\subset L(v,G_0)$ for every $v \in V(G_0)$.

If $\chi(L(v,G_0)) \geqslant \phi(m-1,n)$ for some $v \in V(G_0)$, we get (using the induction hypothesis) the existence of a triangle-free graph H, $H \subset L(v,G_0) \subset G_0$, with $\chi(H) \geqslant n$. Let us suppose now that $\chi(L(v,G_0)) \leqslant \phi(m-1,n) - 1$ for every $v \in V(G_0)$. Let $\bigcup_{i=1}^{k} B_i^v$ be a coloring of the graph $L(v,G_0)$; $k \leqslant \phi(m-1,n) - 1$. Let us define a partition

$$E(G_0) = E_1 \cup E_2 \cup \cdots \cup E_{\phi(m-1,n)-1},$$

$$E_i = \{\{v_i,v\};\, v_i \in B_i^v,\, v \in V(G_0)\}.$$

From Zykov's proposition there exists an i such that $\chi(V,E_i) \geqslant n$. Obviously, (V,E_i) is a triangle-free graph.

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