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DECOMPOSABLE TENSORS AS A QUADRATIC VARIETY

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ABSTRACT. Let V_i be a finite dimensional vector space over a field F for each $i = 1, 2, \ldots, m$, and let z be a tensor in $V_1 \otimes \cdots \otimes V_m$. In this paper a set of homogeneous quadratic polynomials in the coordinates of z is exhibited for which the associated variety is the set of decomposable tensors. In addition, a question concerning the maximal tensor rank in such a situation is answered, and an application to other symmetry classes of tensors is cited.

Introduction. Let V_1 and V_2 be m-dimensional and n-dimensional vector spaces over the filed F. After choosing bases of V_1 and V_2, we may consider $M_{m,n}(F)$, the space of m-by-n matrices over F, to be a model of the tensor space $V_1 \otimes V_2$ (see, for example, [1]). In this model, decomposable tensors correspond to matrices of rank less than two, which is a quadratic variety corresponding to the set of 2-by-2 subdeterminants of $X = [x_{ij}]$, the m-by-n generic matrix of mn indeterminates. In this paper, this result is extended to a tensor product of a finite number of vector spaces.

Notation. Let V_i, $i = 1, \ldots, m$, denote a vector space of dimension n_i over F with ordered basis $E_i = \{e_{i_1}, \ldots, e_{i_{n_i}}\}$, and let $N = n_1 \cdots n_m$.

Let Γ or $\Gamma(n_1, \ldots, n_m)$ denote the set of functions, γ, from $\{1, \ldots, m\}$ to the positive integers which satisfy: $\gamma(i) \leqslant n_i$, $i = 1, \ldots, m$. For $\gamma \in \Gamma$, let $e_\gamma^\otimes = e_{1\gamma(1)} \otimes \cdots \otimes e_{m\gamma(m)}$, so that $E^\otimes = \{e_\gamma^\otimes | \gamma \in \Gamma\}$ is a basis of the tensor product $V_1 \otimes \cdots \otimes V_m$. We shall consider Γ to be ordered via lexicographic ordering.

The N-tuple $(p(\gamma))_{\gamma \in \Gamma}$ will always be associated with the tensor

$$z = \sum_{\gamma \in \Gamma} p(\gamma) e_\gamma^\otimes.$$

For $\alpha, \beta \in \Gamma$, $k = 1, \ldots, m$, let $\alpha[k : \beta]$ denote the sequence obtained by replacing $\alpha(k)$ with $\beta(k)$.

The tensor $x \in V_1 \otimes \cdots \otimes V_m$ will be referred to as *decomposable* if there exist vectors $v_i \in V_i$, $i = 1, \ldots, m$, such that $z = v_1 \otimes \cdots \otimes v_m$.

The main result is the following

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THEOREM. *The tensor* $z = \sum_{\gamma \in \Gamma} p(\gamma) e_\gamma^{\otimes}$ *is decomposable iff*

$$p(\alpha)p(\beta) = p(\alpha[k: \beta])p(\beta[k: \alpha]), \quad k = 1, \ldots, m; \alpha, \beta \in \Gamma.$$

We now will exhibit the necessary lemmas and subsequently the proofs; but first we require some additional notation. For any fixed $t = 1, \ldots, m - 1$, we will say that a tensor z is *t-decomposable* iff there exists $z_1 \in V_1 \otimes \cdots \otimes V_t$, $z_2 \in V_{t+1} \otimes \cdots \otimes V_m$ such that $z = z_1 \otimes z_2$. Also, let Γ_t denote $\Gamma(n_1, \ldots, n_t)$, and let Γ'_t denote $\Gamma(n_{t+1}, \ldots, n_m)$, and for $\gamma \in \Gamma$, let

$$\gamma_t = (\gamma(1), \ldots, \gamma(t)) \in \Gamma_t, \quad \gamma'_t = (\gamma(t+1), \ldots, \gamma(m)) \in \Gamma'_t.$$

Conversely, for $u \in \Gamma_t$, $\nu \in \Gamma'_t$ let $(u, \nu) \in \Gamma$ be defined by

$$(u,\nu) = (u(1), \ldots, u(t), \nu(1), \ldots, \nu(m-t)).$$

Note that $\gamma = (u, \nu)$ iff $\gamma_t = u$ and $\gamma'_t = \nu$.

LEMMA 1. *Z is decomposable iff z is t-decomposable for all* $t = 1, \ldots, m - 1$.

Notation. Let $z(t)$ denote the $n_1 \ldots n_t$-by-$n_{t+1} \ldots n_m$ matrix with rows and columns indexed by Γ_t and Γ'_t, and which has as its entry in the uth row and νth column the value $p((u, \nu))$.

LEMMA 2. *Z is t-decomposable iff*

$$p((u, \nu))p((\alpha, \beta)) = p((\alpha, \nu))p((u, \beta))$$

for all $u, \alpha \in \Gamma_t$; $\nu, \beta \in \Gamma'_t$.

LEMMA 3. *Z is decomposable iff whenever* $\alpha, \beta, \alpha^*, \beta^* \in \Gamma$ *and* $\{\alpha(i), \beta(i)\}$ $(i)\} = \{\alpha^*(i), \beta^*(i)\}$, $i = 1, \ldots, m$, *then* $p(\alpha)p(\beta) = p(\alpha^*)p(\beta^*)$.

PROOF OF LEMMA 1. It suffices to assume z is t-decomposable for all t and show z is decomposable. Since z is 1-decomposable there is a vector $v_1 \in V_1$ and a tensor $z_1 \in V_2 \otimes \cdots \otimes V_m$ such that $z = v_1 \otimes z_1$. Since z is 2-decomposable there exist $z_2 \in V_1 \otimes V_2$ and $z_3 \in V_3 \otimes \cdots \otimes V_m$ such that $v_1 \otimes z_1 = z_2 \otimes z_3$. An easy dimension argument involving the dual space of $V_1 \otimes \cdots \otimes V_m$ (see, for example, [1] for a good background) yields that $z_2 = v_1 \otimes v_2$ for some $v_2 \in V_2$. An induction then completes the proof.

PROOF OF LEMMA 2. In [1] it is shown that the space of $n_1 \ldots n_t$-by-$n_{t-1} \ldots n_m$ matrices is a tensor product of $V_1 \otimes \cdots \otimes V_t$ with $V_{t+1} \otimes \cdots \otimes V_m$, and that in this model the decomposable elements correspond to matrices of rank less than two. These matrices are exactly those whose 2-by-2 subdeterminants all vanish. In the present situation, z is t-decomposable in $V_1 \otimes \cdots \otimes V_m$ iff z is decomposable when viewed as a 2-fold tensor in $(V_1 \otimes \cdots \otimes V_t) \otimes (V_{t+1} \otimes \cdots \otimes V_m)$ iff all the 2-by-2 subdeterminants of $z(t)$ are zero iff the conclusion of Lemma 2 holds.

PROOF OF LEMMA 3. By Lemmas 2 and 3 it suffices to show that the conclusion of Lemma 2 is equivalent to the conclusion of Lemma 3, and it clearly suffices to show that the conclusion of Lemma 2 implies the conclusion of Lemma 3. Assume the conclusion of Lemma 2 holds. For $t = 1$, we have

$$p(\alpha)p(\beta) = p(\alpha[1:\beta])p(\beta[1:\alpha]),$$

and for $t = 2$ we have

$$p(\alpha)p(\beta) = p((\beta_2, \alpha_2'))p((\alpha_2, \beta_2')).$$

Simultaneously, these conditions yield that interchanging the first and/or second elements of α and β will not change the value of $p(\alpha)p(\beta)$. Inductively, if α^*, β^* are formed from α, β by interchanging some of the components, then $p(\alpha)p(\beta) = p(\alpha^*)p(\beta^*)$, and the proof of Lemma 3 is complete.

PROOF OF THE THEOREM. It suffices to show that the conclusion of the theorem implies the conclusion of Lemma 3. This is true since any α^*, β^* can be obtained from α, β by a finite sequence of interchanges of the type explicit in the conclusion of Lemma 3, and the theorem is proved.

A corollary. Let G be a subgroup of S_m, the mth symmetric group, and let $\lambda\colon G \to F$ be a character. Let V be a vector space over F with mth tensor power $\otimes^m V$. Let $T(G, \lambda)$ be the *symmetrizer corresponding to G and* λ, which is a linear map from $\otimes^m V$ onto $V_\lambda(G)$, the *symmetry class of tensors corresponding to G and* λ. The set of decomposable symmetrized tensors, $\{v_1 * \cdots * v_m | v_i \in V_i, i = 1, \ldots, m\}$, is the image of the set of decomposable tensors in $\otimes^m V$, and since the linear image of a quadratic variety is also a quadratic variety, we have proved the following

COROLLARY. *The set of decomposable elements in any symmetry class of tensors is a quadratic variety.*

This result is classically known in the case when $G = S_m$, $\lambda = \varepsilon$, and $V_\lambda(G) = \wedge^m V$, the mth Grassmann space.

EXAMPLE. Let $\{e_1, e_2, e_3, e_4\}$ be a basis of V, and let $z^\otimes \in V \otimes V$ be given by $\sum_{i,j=1}^4 p(i,j)e_i \otimes e_j$. Then the corresponding $z^\wedge$ in $V \wedge V$ is given by $\sum_{1 \leq i < j \leq 4} q(i,j)e_i \wedge e_j$, where $q(i,j) = \frac{1}{2}(p(i,j) - p(j,i))$. It is known that $z^\wedge$ is decomposable in $V \wedge V$ iff the quadratic Plücker relation,

$$q(1,2)q(3,4) + q(2,3)q(1,4) - q(1,3)q(2,4) = 0,$$

is satisifed. By a direct calculation, if the conclusion of the theorem is satisfied for the $p(i,j)$'s, then the quadratic Plücker relation is satisfied for the $q(i,j)$'s.

A conjecture. The idea in the proof of Lemma 2 may be used to answer a conjecture on tensor rank. For a tensor $z \in V_1 \otimes \cdots \otimes V_m$, let $\rho(z)$ be the least positive integer k for which z can be written as a sum of k decomposable tensors. This quantity is referred to as the *tensor rank* of z. Watkins [3] has shown that if $n_1 \geqslant \cdots \geqslant n_m$, then there exist elements of rank n_2; and has asked if this is the maximum rank for elements in $V_1 \otimes \cdots \otimes V_m$.

For $1 \leqslant t < m$ and $1 \leqslant \omega(1) < \cdots < \omega(t) \leqslant m$, let

$$n_\omega = \prod_{i=1}^{t} n_{\omega(i)}, \qquad M = \max_\omega \ \min\{n_\omega, N/n_\omega\}.$$

This quantity also occurs [2] in an inequality giving an upper bound for the dimension of a space W for which there exists an m-linear function,

$$\varphi: V_1 \times \cdots \times V_m \to W,$$

which is onto.

The following answers Watkins' conjecture in the negative.

THEOREM. *There exist tensors of rank at least M in $V_1 \otimes \cdots \otimes V_m$.*

PROOF. We may assume $M = n_1 \cdots n_t$, and we may regard $V_1 \otimes \cdots \otimes V_m$ as a tensor product of the M-dimensional space $V_1 \otimes \cdots \otimes V_t$ with the N/M-dimensional space $V_{t+1} \otimes \cdots \otimes V_m$ by the associativity of tensor product. It is known that in a tensor product of spaces of dimension M and M_1, with $M \leqslant M_1$, there exist tensors of rank M. Hence there exists $z \in V_1 \otimes \cdots \otimes V_m$ which cannot be written as $\sum_{i=1}^{M-1} z_i \otimes z_i'$, with $z_i \in V_1 \otimes \cdots \otimes V_t$, $z_i' \in V_{t+1} \otimes \cdots \otimes V_m$, $i = 1, \ldots, M-1$. The tensor rank of this element is clearly at least M, and the theorem is proved.

Further questions. In relation to Watkins' question, is the M given in the second theorem a maximum for ranks of tensors?

In $V_1 \otimes V_2$, the set of elements of rank less than k forms a kth degree variety (corresponding to the k-by-k subdeterminants of X). Does this fact have an analog when $m > 2$?

What are the analogues of the quadratic Plücker relations in other symmetry classes of tensors? Specifically, what are the relations in the mth symmetric space?

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