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A NOTE ON FLAT ALGEBRAS

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ABSTRACT. The following results are proved. If f: $A \to B$ is a homomorphism of rings, with B noetherian, I is an ideal of B contained in the Jacobson radical, and B/I^n is A-flat, for all n, then f is flat. If, using similar notations and assumptions, I is generated by a regular sequence, then the flatness of B/I implies the flatness of f. A simple geometric application is given.

In this note we present some results on flat homomorphisms of rings. They are very basic but, apparently, they did not receive the attention of researchers in this area.

In the sequel, all rings are commutative, with identity, and homomorphisms respect the identities. The Jacobson radical of a ring B is denoted by $\operatorname{rad}(B)$.

THEOREM. *Let f: $A \to B$ be a homomorphism of rings, with B noetherian, I an ideal of B, contained in* $\operatorname{rad}(B)$. *Assume the induced homomorphism f_n: $A \to B/I^n$ is flat, for all n; then, f is flat.*

PROOF. The following facts are well known: (a) The flatness of f is equivalent to the following: for any finitely generated ideal J of A, the canonical map $J \otimes_A B \to B$ is injective (cf. [1, p. 18]). (b) If J is a finitely generated ideal of A, then the B-module $J \otimes_A B$ is separated in the I-adic topology (see [1, Example 1, p. 145]). Hence, to prove our Theorem it suffices to show: Given a finitely generated ideal J of A, if h: $J \otimes_A B \to B$ is the canonical map, $K = \operatorname{Ker}(h)$, and n any positive integer, then

$$K \subset I^n (J \otimes_A B).$$

To show this, note that we have canonical isomorphisms:

$$(1) \quad J \otimes_A (B/I^n) = (J \otimes_A B) \otimes_B (B/I^n) = (J \otimes_A B)/I^n (J \otimes_A B),$$

$$(2) \quad J \otimes_A I^n = I^n (J \otimes_A B).$$

The first is clear. To see the second, note that the exact sequence of A-modules,

$$0 \to I^n \to B \to B/I^n \to 0,$$

induces, by the A-flatness of B/I^n, an exact sequence

$$0 \to J \otimes_A I^n \to J \otimes_A B \xrightarrow{a} J \otimes_A (B/I^n) \to 0.$$

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By (1), a can be identified to the canonical map

$$b\colon J \otimes_A B \to (J \otimes_A B)/I^n(J \otimes_A B);$$

hence, $J \otimes_A I^n$ is canonically isomorphic to $\operatorname{Ker}(b) = I^n(J \otimes_A B)$.

Using (1) and (2), we get a commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & I^n(J \otimes_A B) & & \\
 & & & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & J \otimes_A B & \xrightarrow{h} & B \\
 & & \downarrow & & \downarrow{\scriptstyle c} & & \downarrow \\
 & & K \otimes_B (B/I^n) & \xrightarrow{d} & J \otimes_A (B/I^n) & \xrightarrow{b} & B/I^n \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

where the column and first row are exact and $bd = 0$. Moreover, b is the canonical map, which is injective because B/I^n is A-flat. Hence, $c(K) \subset \operatorname{Ker}(b) = 0$; it follows that $K \subset \operatorname{Ker}(c) = I^n(J \otimes_A B)$ as we wanted. The Theorem is proved.

COROLLARY 1. *Let A, B and f be as in the Theorem. Let $b_1, \ldots, b_n$ be elements of B which form a regular sequence (cf. [1, p. 95]). Assume $I = (b_1, \ldots, b_r)B \subset \operatorname{rad}(B)$ and that B/I is A-flat. Then, f is flat.*

PROOF. It is well known that our assumptions on I imply that the B/I-module I^n/I^{n+1} is projective (hence, flat) for any positive integer n. Since B/I is A-flat, I^n/I^{n+1} is A-flat, for all n. Then, by using the exact sequences

$$0 \to I^n/I^{n+1} \to B/I^{n+1} \to B/I^n \to 0$$

and induction, we see that the A-module B/I^n is flat, for all n. Now we use the Theorem to conclude that B is A-flat.

The following geometric application is an immediate consequence of Corollary 1.

COROLLARY 2. *Let $g\colon X \to Y$ be a morphism of schemes (with X locally noetherian), Z a closed subscheme of X, defined by a sheaf of ideals $\mathfrak{I}$, such that $\mathfrak{I}_x$ is generated by a regular sequence, for all x in X. Assume that the restriction of g to $X - Z$ and the composition $Z \subset X \to^g Y$ are flat. Then, g is flat.*

REMARK. 1. The conclusion of Corollary 1 does not hold if the assumption "$b_1, \ldots, b_r$ is a regular sequence" is removed. For instance, if $B = k[[x,y]]/(x,y^2)$, $b_1 = y$, $A = k[[x]]$, where k is a field, then $B/(y)B$ is A-flat, but B is not.

REMARK 2. Probably the most interesting case of Corollary 2 is when Z is locally defined by a single equation, i.e., when Z is a Cartier divisor. In this case, Corollary 2 says that the usual requirements for a relative Cartier divisor can be somewhat weakened (see [2, Definition 21.15.2]). Actually, in this case the condition "X locally noetherian" can be eliminated, as the following argument (which was communicated to us by Michael Artin) shows. Algebraically, we are proving: Given a homomorphism $f: A \to B$ (two arbitrary rings), $b \in B$ a non-zero-divisor, if $B/(b)B$ and the ring of fractions B_b are A-flat, then f is flat. We may assume that b is not a unit. Consider the exact sequences:

$$S_n: 0 \to B \xrightarrow{b^n} B \to B/(b^n) \to 0.$$

Multiplication times b induces homomorphisms $S_n \to S_{n+1}$ (with id: $B \to B$ for all n). Thus, we get a direct system of exact sequences, whose direct limit is the exact sequence:

$$0 \to B \to B_b \to \operatorname{ind\,lim} B/(b^n)B \to 0. \tag{3}$$

As in the proof of Corollary 2, $B/(b^n)B$ is A-flat, for all n. Since ind lim preserves flatness, the second and third terms of (3) are flat, hence B is A-flat.

Note 1. An argument similar to that of Remark 2 appears as Lemma (1.4.2.1) in Part 2 of [3].

Note 2. We want to thank the referee, who suggested simplifications of the proofs of the results above, and pointed out the content of Note 1.

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