PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 64, Number 2, June 1977

DUALITY AND ALTERNATIVE IN MULTIOBJECTIVE OPTIMIZATION

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ABSTRACT. Within a general framework containing multiobjective optimization, an equivalence between duality properties and alternative conditions is established for pairs of constrained optimization problems. Sufficient conditions for a Pareto type duality and a multiobjective strong duality are obtained.

1. **Introduction.** In a recent paper, McLinden [2] presented in the case of scalar optimization a general equivalence between duality and alternative, which underlies various known specific instances. The present paper extends McLinden's approach to the case of multiobjective optimization. Moreover, the multiobjective functions are replaced by a somewhat more general framework involving "level sets". Concerning the way one can associate to a given primal multiobjective optimization problem a convenient dual one, the "conjugate" construction in [4] and [1], extending Fenchel's approach, can be used. By this means, the main result in [4] (Theorem 2) can be seen to fall within the general framework presented in this paper.

2. **Primal and dual constrained multiobjective optimization problems.** Assume the partially ordered set $(V, \leqslant)$, $V \neq \varnothing$, represents the multiobjective valuation. The primal problem is defined on the set of "constraints" X, by a primal family of "level sets" $\mathfrak{X} = (X_v | v \in V)$ satisfying the condition: $X_v \subset X_{v'} \subset X$ for $v, v' \in V$, $v \leqslant v'$. The dual problem is defined on the set of "constraints" Y, by a dual family of "level sets" $\mathfrak{Y} = (Y_v | v \in V)$ satisfying the dual condition: $Y_{v'} \subset Y_v \subset Y$ for v, $v' \in V$, $v \leqslant v'$. Denote $V_P = \{v \in V | X_v \neq \varnothing\}$ and $V_D = \{v \in V | Y_v \neq \varnothing\}$. A first pair of primal and dual problems, corresponding to the usual multiobjective optimization (see (5)) is given by:

$$\inf V_P = ?, \qquad \sup V_D = ? \,. \tag{1}$$

Denote by W_P the set of all minimal elements u in V_P such that $u = \inf U$ for a certain $U \subset V_P$, $U \neq \varnothing$ and U totally ordered. Similarly, denote by W_D the set of all maximal elements u in V_D such that $u = \sup U$ for a certain $U \subset V_D$, $U \neq \varnothing$ and U totally ordered. A second pair of primal and dual problems, corresponding to Pareto optimization (see (6)) is given by:

Received by the editors March 12, 1976 and, in revised form, December 6, 1976.
AMS (*MOS*) *subject classifications* (1970). Primary 90C99; Secondary 06A10, 90C25, 90C30.
Key words and phrases. Dual multiobjective problems, alternative conditions, Pareto duality.

$$(2) \qquad W_P = ?, \qquad W_D = ?\,.$$

In the presence of primal and dual multiobjective functions $f\colon X \to V$, $g\colon Y \to V$ (cf. [2], [4]), the level sets can be considered given by:

$$(3) \quad X_v = \{x \in X \mid f(x) \leqslant_{\neq} v\}, \qquad Y_v = \{y \in Y \mid v \leqslant g(y)\} \quad \text{for } v \in V.$$

In case f satisfies the condition:

$$(4) \qquad \forall x \in X\colon \exists v \in V\colon f(x) \leqslant_{\neq} v$$

that is, none of the values taken by f is maximal in V, the problems in (1) and (2) obtain the following usual interpretations.

First, suppose $(V, \leqslant)$ is dense and for any $v \in V$ with $(v, \to] =^{\triangle} \{v' \in V \mid v \leqslant_{\neq} v'\}$ nonempty, $\inf(v, \to]$ exists. Then, (1) is equivalent to

$$(5) \qquad \inf f(X) = ?, \qquad \sup g(Y) = ?\,.$$

That is, $\inf V_P$ (resp. $\sup V_D$) exists if and only if $\inf f(X)$ (resp. $\sup g(Y)$) exists and they are equal.

Next, suppose $(V, \leqslant)$ is dense and any nonempty, totally ordered, lower bounded subset in V has an infimum. Then

$$(6) \quad \text{Pareto} \inf f(X) = f(X) \backslash V_P \subset W_P, \qquad \text{Pareto} \sup g(Y) = W_D \cap V_D.$$

REMARK. The level set formulation cannot always be given a representation under the form of (3). For example, let $V = [0, 1] \times [0, 1]$, take the product order on V, let $X = Y = V$ and let $X_v = \{x = (x_1, x_2) \in X \mid x_1 + x_2 < v_1 + v_2\}$ and $Y_v = \{y = (y_1, y_2) \in Y \mid v_1 + v_2 \leqslant y_1 + y_2\}$ for $v = (v_1, v_2) \in V$.

3. **Equivalent alternative conditions and duality properties.** Two alternative conditions are considered:

(AW) $V_P \cap V_D = \varnothing$ ("weak alternative"),

(AS) $V_P \cup V_D = V$ ("strong alternative").

The joint condition (AW) and (AS), denoted (A), is called "alternative".

The two corresponding duality properties are:

(DW) $\forall v \in V_P,\ w \in V_D\colon \operatorname{non}(v \leqslant w)$ ("weak duality"),

(DS) $\forall v \in V \backslash V_P,\ w \in V \backslash V_D\colon \operatorname{non}(w \leqslant v)$ ("strong duality").

The joint property (DW) and (DS), denoted (D), is called "duality".

Suppose the level sets are given by (3). Then (DW) is equivalent to the condition:

$$\forall x \in X, y \in Y\colon \operatorname{non}(f(x) \leqslant_{\neq} g(y)).$$

If $\sup g(Y) \leqslant \inf f(X)$, then (DS) implies equality in that relation.

A general equivalence between these notions of duality and alternative can be proved, extending [2].

THEOREM 1. (A) *and* (D) *are equivalent.*

PROOF. (AW) $\Rightarrow$ (DW). Suppose $v \in V_P$, $w \in V_D$, $v < w$. Then $w \in V_P$, since $v \in V_P$ and $v \leqslant w$. Therefore, $w \in V_P \cap V_D$, contradicting (AW). (DW) $\Rightarrow$ (AW). Suppose $v \in V_P \cap V_D$, then (DW) implies non($v < v$), contradicting the reflexivity of $<$. (AS) $\Rightarrow$ (DS). Suppose $v \in V \setminus V_P$, $w \in V \setminus V_D$, $w < v$. Then $w \in V \setminus V_P$, since $w \in V_P$, $w < v$ imply $v \in V_P$. Therefore, $w \in V \setminus (V_P \cup V_D)$, contradicting (AS). (DS) $\Rightarrow$ (AS). Suppose

$$v \in V \setminus (V_P \cup V_D),$$

then (DS) implies non($v \leqslant v$), contradicting the reflexivity of $<$.

We consider now a Pareto type duality property (see Corollary 1):

(DP) $W_P \cap V_P \subset W_D$, $W_D \cap V_D \subset W_P$.

The partial order $(V, \leqslant)$ is called "chain complete", if and only if each nonempty, totally ordered, lower or upper bounded subset in V has an infimum, respectively supremum.

The next result gives a general sufficient condition for (DP).

THEOREM 2. *Suppose $(V, \leqslant)$ is dense and chain complete, and suppose the following conditions hold*:

$$\text{(7)} \qquad \forall v \in V_P\colon \exists w \in V_D\colon w \leqslant v \quad \textit{and} \quad \forall w \in V_D\colon \exists v \in V_P\colon w \leqslant v.$$

Then (A), *and therefore* (D), *implies* (DP).

PROOF. First we prove the inclusion $W_P \cap V_P \subset W_D$. Suppose $u \in W_P \cap V_P$, and let $V_1 = \{v_1 \in V_D | v_1 \leqslant u\}$. Then $V_1 \neq \varnothing$, due to (7). The family of nonempty, totally ordered subsets of V_1 satisfies Zorn's lemma. Suppose U is maximal in that family. Then $u_1 = \sup U$ exists and $u_1 < u$. Suppose $u_1 \leqslant_{\neq} u$. Then $\exists u' \in V\colon u_1 \leqslant_{\neq} u' \leqslant_{\neq} u$, since $(V, <)$ is dense. Moreover, $u' \notin V_D$. Indeed, $u' \in V_D$, $u' \leqslant_{\neq} u$ imply $u' \in V_1$. Let $U' = U \cup \{u'\}$. Then $\sup U = u_1 \leqslant_{\neq} u' \in V_1$ imply $U \subset_{\neq} U' \subset V_1$, $U' \neq \varnothing$, U' totally ordered, contradicting the maximality of U. Now $u' \notin V_D$ implies $u' \in V_P$ because of (AS). But the relations $u' \leqslant_{\neq} u$, $u' \in V_P$ contradict $u \in W_P$. Therefore $u = \sup U$. In order to prove that $u \in W_D$, it remains to show that $v \in V_D$, $u < v$ imply $v = u$. But the situation $v \in V_D$, $u < v$ cannot occur, since it would imply $u \in V_D$, which, together with the hypothesis $u \in W_P \cap V_P \subset V_P$, would contradict (AW).

The inclusion $W_D \cap V_D \subset W_P$ follows in a similar way.

COROLLARY 1. *Assume the conditions in Theorem* 2 *are satisfied and that the level sets are given by* (3). *Then* (A), *and therefore* (D), *implies*

$$\text{(8)} \qquad \text{Pareto inf} f(X) \subset \text{Pareto sup} g(Y).$$

PROOF. In view of (AS), the relation (see (6)) Pareto $\inf f(X) = f(X) \setminus V_P$ implies that Pareto $\inf f(X) \subset V_D$. Further, (A) implies Pareto $\inf f(X) \subset W_D$. Indeed, suppose $v \in$ Pareto $\inf f(X)$. Then $\exists x \in X\colon v = f(x) \notin V_P$, and hence $v \in V_D$, by virtue of (AS). Let $U = \{v\}$. Then $U \subset V_D$, $U \neq \varnothing$, U

totally ordered and $v = \sup U$. Thus, in order to prove that $v \in W_D$, it remains to show that $v' \in V_D$, $v \leqslant v'$ imply $v' = v$. Suppose $v \leqslant_{\neq} v'$. Then $x \in X_{v'}$, since $f(x) = v \leqslant_{\neq} v'$. Thus $v' \in V_P$. But this together with $v' \in V_D$ contradicts (AW). This completes the proof, since Pareto $\sup g(Y) = W_D \cap V_D$ according to (6).

A sufficient condition for a multiobjective strong duality property is now available.

COROLLARY 2. *Assume the conditions in Corollary* 1. *Then* (A), *and therefore* (D), *implies that* $\inf f(X) \leqslant \sup g(Y)$, *whenever both terms exist.*

PROOF. It results from (8) and the obvious inclusions Pareto $\inf f(X) \subset f(X)$ and Pareto $\sup g(Y) \subset g(Y)$.

4. **Simplifying the multiobjective valuation.** The pairs of primal and dual multiobjective optimization problems in (1) and (2) are perfectly defined by:

$$\mathcal{O} = ((V, \leqslant), X, Y, \mathcal{X}, \mathcal{Y}) \tag{9}$$

where $(V, \leqslant)$ is the multiobjective valuation, X and Y are respectively the primal and dual set of "constraints", and finally $\mathcal{X} = (X_v | v \in V)$ and $\mathcal{Y} = (Y_v | v \in V)$ are respectively the primal and dual "level set" families. In view of that fact, we shall in the following identify (9) with the problems in (1) and (2).

We now give a method of replacing the initial multiobjective optimization problem $\mathcal{O} = ((V, \leqslant), X, Y, \mathcal{X}, \mathcal{Y})$ by an equivalent one $\mathcal{O}_* = ((V_*, \leqslant_*), X, Y, \mathcal{X}_*, \mathcal{Y}_*)$ which will always satisfy (AS). Moreover, $\mathcal{O}_*$ will satisfy (AW) if and only if $\mathcal{O}$ satisfies that condition. The method consists in eliminating from V those elements which do not participate in the optimization. To this end, define the parial order $(V_*, \leqslant_*)$ by setting $V_* = V_P \cup V_D$ and letting $\leqslant_*$ be the restriction of $\leqslant$ to V_*. Obviously $V_* = \varnothing$ if and only if $\forall v \in V$: $X_v = Y_v = \varnothing$ in which case the initial problem itself is trivial. Therefore, we shall assume $V_* \neq \varnothing$. Consider the families of level sets $\mathcal{X}_* = (X_v | v \in V_*)$ and $\mathcal{Y}_* = (Y_v | v \in V_*)$. Obviously

$$V_{*P} = V_P, \qquad V_{*D} = V_D. \tag{10}$$

Therefore, the problems in (1) are the same for both $\mathcal{O}_*$ and $\mathcal{O}$ if and only if $\mathcal{O}$ satisfies

$$\inf_{\leqslant} V_P, \qquad \sup_{\leqslant} V_D \in V_P \cup V_D. \tag{11}$$

Further, due to (10) one obtains

$$W_P \cap V_* \subset W_{*P} \subset V_*, \qquad W_D \cap V_* \subset W_{*D} \subset V_*.$$

Therefore, the problems in (2) are the same for both $\mathcal{O}_*$ and $\mathcal{O}$ if and only if

$$W_P \cup W_D \subset V_P \cup V_D, \qquad W_{*P} \subset W_P, \qquad W_{*D} \subset W_D.$$

The latter condition can be simplified as follows.

THEOREM 3. *Suppose* $(V, \leqslant)$ *is chain complete. Then the problems in* (2) *are the same for both* $\mathcal{O}_*$ *and* $\mathcal{O}$ *if and only if* $\mathcal{O}$ *satisfies*

$$W_P \cup W_D \subset V_P \cup V_D. \tag{12}$$

PROOF. It suffices to show that (12) implies the inclusions $W_{*P} \subset W_P$ and $W_{*D} \subset W_D$. Suppose $u_* \in W_{*P}$. Then $u_* = \inf_{\leqslant_*} U_*$ with $U_* \subset V_{*P}$, $U_* \neq \varnothing$ and U_* totally ordered by $\leqslant_*$. Since $u_* \in W_{*P} \subset V$ is a lower bound of U_* in $(V, \leqslant)$ it follows that $u = \inf_{\leqslant} U_*$ exists and $u_* \leqslant u$. If $u \in W_P$ then $u \in V_*$ follows by (12). Therefore $u = u_*$ and then $u_* \in W_P$. Suppose $u \notin W_P$. But $u = \inf_{\leqslant_*} U_*$, $U_* \subset V_{*P} = V_P$, $U_* \neq \varnothing$ and U_* totally ordered in $(V, \leqslant)$. Thus, $\exists v \in V_P$: $v \leqslant_{\neq} u$ according to the definition of W_P (§2). Then $\forall u' \in U_*$: $v \leqslant_{\neq} u'$, hence $v \leqslant u_*$, since $v \in V_P \subset V_*$. But $v \leqslant_* u_*$, $v \in V_P = V_{*P}$ imply $v = u_*$, since $u_* \in W_{*P}$. Finally, $u_* \in V_P$. Now $u_* \leqslant u$ implies $u \in V_P$. But then actually $u = \inf_{\leqslant_*} U_*$, since $u \in V_P \subset V_*$. We can conclude that $u = u_*$. Therefore, $v \leqslant_{\neq} u$ implies $v \leqslant_{\neq} u_*$, contradicting $v = u_*$. The inclusion $W_{*D} \subset W_D$ is proved in a similar way.

REMARK. Suppose that the multiobjective optimization problem $\mathcal{O} = ((V, \leqslant), X, Y, \mathfrak{X}, \mathfrak{Y})$ is defined as in (3) by primal f: $X \to V$ and dual g: $Y \to V$ multiobjective functions obtained by the "conjugate" construction in [4] (see also [1]). Then ([4, Lemma 1])

$$\forall x \in X, y \in Y: g(y) \leqslant f(x) \tag{13}$$

so that $\mathcal{O}$ satisfies (AW). Moreover, within the construction in [4] the partial order $(V, \leqslant)$ is dense and chain complete and f satisfies (4). Therefore, (5) and (1) are identical in that context.

Now passing from $\mathcal{O}$ to $\mathcal{O}_*$, we get that $\mathcal{O}_*$ satisfies (A) and therefore (D). The condition (11) will be necessary and sufficient for (5) to be equivalent to (1) for $\mathcal{O}_*$. But obviously $\inf f(X) \notin V_P$, while $\sup g(Y) \notin V_P$ because of (13). Therefore, (11) is equivalent to

$$\inf f(X) \in V_D \quad \text{and} \quad \sup g(Y) \in V_D.$$

Now, the first of these implies $\inf f(X) \leqslant \sup g(Y)$, while the second is equivalent to $\sup g(Y) = \max g(Y)$. It follows that the main result in [4] (Theorem 2) is tantamount to proving (11).

ACKNOWLEDGEMENT. The author is very glad to express his special gratitude to the Referee for the numerous suggestions. The suggestion of possible connections with [4] and [1] resulted in the Remark in §4.

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