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A NEW CHARACTERIZATION OF BARRELLED SPACES

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ABSTRACT. A locally convex space E is barrelled if and only if $L_s(E, F)$ is quasicomplete for every Banach space F.

Recently A. Wilansky [5] has shown that a locally convex space E is barrelled if for every compact Hausdorff space X and linear map $T\colon E \to \mathcal{C}(X)$ with closed graph, T is continuous. His proof runs as follows: For any barrel B in E, endow its absolute polar $H := B^\circ$ with the topology induced by the weak topology $\sigma(E', E)$, form the Banach space $F := \mathcal{C}^*(H)$ of bounded continuous functions on H (which can be identified with $\mathcal{C}(\beta H)$), and show that the linear map $T\colon E \to F$, $T(x)(h) := h(x)$ for $x \in E$, $h \in H$, has closed graph. By hypothesis the continuity of T follows, and so B, which is the inverse image of the unit disc in F, is a neighborhood of 0.

Now let $L_s(E, F)$ be the space of continuous linear maps of E into F, s indicating the topology of pointwise convergence. We shall show that the continuity of T can be inferred as well, if quasicompleteness of $L_s(E, F)$ is supposed. To this end we shall define for every finite set $A \subset E$ and $\varepsilon > 0$ a linear map $T_{A,\varepsilon} \in L(E, F)$ such that

$$\sup\{\|T_{A,\varepsilon}(x)\|;\ A \subset E \text{ finite}, \varepsilon > 0\} < \infty \quad \text{for } x \in E, \text{ and} \tag{1}$$

$$\|T_{A,\varepsilon}(x) - T(x)\| < \varepsilon \quad \text{for } x \in A. \tag{2}$$

Then from (1), (2) and our hypothesis, the continuity of T will be clear.

Let d be the pseudo-metric defined by

$$d(h, h') := \sup\{|h(x) - h'(x)|;\ x \in A\} \quad \text{on } H.$$

Then the uniformity of (H, d) is obviously coarser than the uniformity induced by $\sigma(E', E)$ on H, so (H, d) is precompact. Therefore, there exist $h_1, \ldots, h_n \in H$ such that $H \subset \bigcup_{i=1}^n K(h_i; \varepsilon)$ with $K(h_i;\ \varepsilon) := \{h \in H;\ d(h_i, h) < \varepsilon\}$. From [3, Corollary 5.35 and Problem 5.W] applied to (H, d) and $(K(h_i, \varepsilon);\ i = 1, \ldots, n)$, we get $\varphi_i \in \mathcal{C}^*(H)$, $i = 1, \ldots, n$, with

$$0 \leqslant \varphi_i \leqslant 1, \quad \sum_{i=1}^n \varphi_i = 1, \quad \operatorname{supp} \varphi_i \subset K(h_i; \varepsilon). \tag{3}$$

Now let

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$$T_{A,\varepsilon}(x)(h) := \sum_{i=1}^{n} \varphi_i(h)h_i(x) \quad \text{for } x \in E, h \in H.$$

Then obviously $T_{A,\varepsilon} \in L(E, F)$, and using (3) one can easily verify that (1) and (2) are satisfied.

Since $L_s(E, F)$ is quasicomplete if E is barrelled and F is any quasicomplete locally convex space [1, Chapter III, §3.7, Corollary 2], we obtain the following characterization of barrelledness:

THEOREM. *A locally convex space E is barrelled if and only if $L_s(E, F)$ is quasicomplete for every Banach space F.*

A more refined version of this theorem involving cardinals will appear in [4, Satz 2.2]. As a corollary of this, one obtains a characterization of those locally convex spaces E for which Kalton's closed graph theorem [2, Theorem 2.6] is valid by the condition that $L_s(E, F)$ is sequentially complete for every separable Banach space F.

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