A TOPOLOGICAL VERSION OF ◊

JOHN GINSBURG

Abstract. The set-theoretic principle ◊ is shown to be equivalent to the existence of universal \( \omega_1 \)-sequences in certain topological spaces. An \( \omega_1 \)-sequence \( (x_\alpha : \alpha \in \omega_1) \) in a space \( X \) is said to be universal in \( X \) if for every point \( p \in X \) there is a stationary set \( S \subseteq \omega_1 \) so that the net \( (x_\alpha : \alpha \in S) \) converges to \( p \). It is shown that the existence of universal \( \omega_1 \)-sequences in spaces of weight \( \omega < c \) is equivalent to ◊.

1. Preliminaries. The set-theoretic principle ◊, due to Jensen [2], has found many interesting applications in topology, for example the construction of Souslin lines [2] and various \( S \)-spaces [1], [4]. In this note we give a version of ◊ in more topological terms.

Our set-theoretic notation and terminology are standard. Recall that a subset \( C \) of \( \omega_1 \) is c.u.b. if it is uncountable and closed in the order topology on \( \omega_1 \), and a subset \( S \subseteq \omega_1 \) is stationary if it intersects all c.u.b. sets. Jensen's principle ◊ is the following statement:

◊: There is a sequence \( (S_\alpha : \alpha \in \omega_1) \) of subsets of \( \omega_1 \) such that for every \( \alpha \in \omega_1 \), \( S_\alpha \subseteq \alpha \) and for every subset \( S \) of \( \omega_1 \), \( \{ \alpha \in \omega_1 : S \cap \alpha = S_\alpha \} \) is stationary.

We will use the following basic fact on stationary sets, for which we refer the reader to Theorem 2.3 in [3].

1.1. Pressing-down lemma. Suppose \( S \subseteq \omega_1 \) is stationary and \( f : S \to \omega_1 \) such that for all \( \alpha \in S \), \( f(\alpha) < \alpha \). Then for some \( \beta \in \omega_1 \), \( f^{-1}(\beta) \) is stationary.

Our principal topological reference is [6]. We will be using two particular topological spaces in our discussion; \( BN - N \) (the remainder of the Stone-Čech compactification of the integers), and \( 2^{\omega_1} \) with the usual product topology. \( BN - N \) has weight \( c \). \( 2^{\omega_1} \) has weight \( \omega_1 \). We recall the following result, due to Parovičenko.

1.2. Theorem [5]. Every compact space having weight at most \( \omega_1 \) is the continuous image of \( BN - N \).

Our version of ◊ involves convergence. Recall that, if \( (D, <) \) is a directed set, and if \( (x_\alpha : \alpha \in D) \) is a net in the space \( X \) directed by \( D \), and if \( p \in X \),
then \((x_a; \alpha \in D)\) converges to \(p\) if, for every neighborhood \(G\) of \(p\) in \(X\) there is \(\alpha_0 \in D\) such that if \(\alpha \in D\) and \(\alpha > \alpha_0\), then \(x_\alpha \in G\). The central concept in our theorem is the notion of a universal \(\omega_1\)-sequence. Let \(X\) be a space and let \((x_\alpha; \alpha \in \omega_1)\) be an \(\omega_1\)-sequence in \(X\). We say \((x_\alpha; \alpha \in \omega_1)\) is universal in \(X\) if for every point \(p \in X\) there is a stationary set \(S \subseteq \omega_1\) so that the subnet \((x_\alpha; \alpha \in S)\) converges to \(p\).

2. \(\Diamond\) and universal \(\omega_1\)-sequences. We now give a version of \(\Diamond\) in topological terms.

2.1. Theorem. \(\Diamond\) is equivalent to the statement that any of the following spaces contains a universal \(\omega_1\)-sequence:

(i) \(\beta N - N\),
(ii) \(2^{\omega_1}\),
(iii) all spaces of weight \(\leq \omega_1\),
(iv) all spaces of weight \(\leq c\).

Proof. \(\Diamond \rightarrow (iv)\): Assume that \((S_\alpha; \alpha \in \omega_1)\) satisfies \(\Diamond\). As is well known, \(\Diamond \rightarrow CH\). (Clearly every countable subset of \(\omega_1\) occurs among the \(S_\alpha\), and so \(2^{\omega_1} = \omega_1\).) Now let \(X\) be a space of weight \(\leq 2^{\omega_1} = \omega_1\); we can choose a basis \(B = \{G_\alpha; \alpha < \omega_1\}\) for \(X\) indexed by \(\omega_1\). We define the sequence \((x_\alpha; \alpha \in \omega_1)\) as follows: if \(\bigcap_{\beta \in S_\alpha} G_\beta \neq 0\), let \(x_\alpha\) be any point in \(\bigcap_{\beta \in S_\alpha} G_\beta\), and otherwise let \(x_\alpha\) be an arbitrary point of \(X\). We claim that \((x_\alpha; \alpha \in \omega_1)\) is universal in \(X\). For, let \(p \in X\). Let \(A = \{\alpha \in \omega_1; p \in G_\alpha\}\). Then by \(\Diamond\), the set \(S = \{\alpha \in \omega_1; A \cap \alpha = S_\alpha\}\) is stationary in \(\omega_1\). We now show that the subnet \((x_\alpha; \alpha \in S)\) converges to \(p\). Let \(N\) be a neighborhood of \(p\) in \(X\). Since \(B\) is a basis, there is an \(\alpha_0 \in \omega_1\) such that \(p \in G_{\alpha_0} \subseteq N\). Suppose \(\alpha \in S\) and \(\alpha > \alpha_0\). Then \(S_\alpha = A \cap \alpha\), so \(S_\alpha = \{\beta \in \alpha; p \in G_\beta\}\). Thus \(\alpha_0 \in S_\alpha\), and \(p \in \bigcap_{\beta \in S_\alpha} G_\beta\). Since \(\bigcap_{\beta \in S_\alpha} G_\beta \neq 0\), we have \(x_\alpha \in \bigcap_{\beta \in S_\alpha} G_\beta\). In particular, \(x_\alpha \in G_{\alpha_0} \subseteq N\). Thus \(\alpha \in S\) and \(\alpha > \alpha_0\) implies \(x_\alpha \in N\). Therefore \((x_\alpha; \alpha \in S)\) converges to \(p\).

This shows \((x_\alpha; \alpha \in \omega_1)\) is universal in \(X\).

(iv) \(\rightarrow (i)\): Trivial.

(i) \(\rightarrow (ii)\): By 1.2, \(2^{\omega_1}\) is the continuous image of \(\beta N - N\). Noting that a continuous map takes a universal sequence onto a universal sequence completes the proof of this implication.

(ii) \(\rightarrow \Diamond\): Let \((f_\alpha; \alpha \in \omega_1)\) be universal in \(2^{\omega_1}\). We will show that for every \(f \in 2^{\omega_1}\), \(\{\alpha \in \omega_1; f|\alpha = f_\alpha|\alpha\}\) is stationary. There is a stationary set \(S\) in \(\omega_1\) so that \((f_\alpha; \alpha \in S)\) converges to \(f\) in \(2^{\omega_1}\). Let \(T = \{\alpha \in S; f|\alpha \neq f|\alpha\}\). We claim that \(T\) is nonstationary. For, suppose \(T\) is stationary. For \(\alpha \in T\), let \(x_\alpha\) be the first ordinal at which \(f_\alpha\) and \(f\) differ. Then \(x_\alpha < \alpha\) for all \(\alpha \in T\), and so the function \(\alpha \rightarrow x_\alpha\) is pressing down on \(T\). By 1.1, there is a stationary set \(T_1 \subseteq T\) and an ordinal \(\beta \in \omega_1\) so that \(x_\alpha = \beta\) for all \(\alpha \in T_1\). Thus \(f_\alpha(\beta) \neq f(\beta)\) for all \(\alpha \in T_1\). But \(\{g \in 2^{\omega_1}; g(\beta) = f(\beta)\}\) is a neighborhood of \(f\) in \(2^{\omega_1}\). Since \((f_\alpha; \alpha \in S)\) converges to \(f\), this neighborhood contains all but countably many of the \(f_\alpha\), \(\alpha \in S\). But \(T_1\) is uncountable, and the neighborhood contains none of the \(f_\alpha\), \(\alpha \in T\). This is a contradiction. Therefore \(T\)
is indeed nonstationary. Since $S$ is stationary, $S - T$ is stationary, and so the set \( \{ \alpha \in \omega_1 : f|_\alpha = f|_\alpha \} \), which contains $S - T$, is also stationary. We have shown that for every $f \in 2^\omega_1$, \( \{ \alpha \in \omega_1 : f|_\alpha = f|_\alpha \} \) is stationary. Using the correspondence between a set and its characteristic function, it is easy to see that setting \( S_\alpha = f_\alpha^{-1}\{1\} \cap \alpha \) gives a sequence \( (S_\alpha : \alpha \in \omega_1) \) which satisfies $\diamondsuit$.

Since, trivially, $\text{(iv)} \rightarrow \text{(iii)} \rightarrow \text{(ii)}$, this completes the proof.

We remark that various other topological spaces can replace $\beta N - \mathbb{N}$ or $2^{\omega_1}$ in the preceding equivalence. Thus, for example, the existence of universal $\omega_1$-sequences in $\beta N$ or in $2^{\omega_1}$ with the lexicographic order topology is equivalent to $\diamondsuit$.

REFERENCES


Department of Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada