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A TOPOLOGICAL VERSION OF $\diamond$

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ABSTRACT. The set-theoretic principle $\diamond$ is shown to be equivalent to the existence of universal ω_1-sequences in certain topological spaces. An ω_1-sequence $(x_\alpha: \alpha \in \omega_1)$ in a space X is said to be *universal in X* if for every point $p \in X$ there is a stationary set $S \subseteq \omega_1$ so that the net $(x_\alpha: \alpha \in S)$ converges to p. It is shown that the existence of universal ω_1-sequences in spaces of weight $< c$ is equivalent to $\diamond$.

1. **Preliminaries.** The set-theoretic principle $\diamond$, due to Jensen [2], has found many interesting applications in topology, for example the construction of Souslin lines [2] and various S-spaces [1], [4]. In this note we give a version of $\diamond$ in more topological terms.

Our set-theoretic notation and terminology are standard. Recall that a subset C of ω_1 is *c.u.b.* if it is uncountable and closed in the order topology on ω_1, and a subset $S \subseteq \omega_1$ is *stationary* if it intersects all c.u.b. sets. Jensen's principle $\diamond$ is the following statement:

$\diamond$: *There is a sequence* $(S_\alpha: \alpha \in \omega_1)$ *of subsets of* ω_1 *such that for every* $\alpha \in \omega_1$, $S_\alpha \subseteq \alpha$ *and for every subset* S *of* ω_1, $\{\alpha \in \omega_1: S \cap \alpha = S_\alpha\}$ *is stationary.*

We will use the following basic fact on stationary sets, for which we refer the reader to Theorem 2.3 in [3].

1.1. PRESSING-DOWN LEMMA. *Suppose* $S \subseteq \omega_1$ *is stationary and* $f: S \to \omega_1$ *such that for all* $\alpha \in S, f(\alpha) < \alpha$. *Then for some* $\beta \in \omega_1, f^{-1}\{\beta\}$ *is stationary.*

Our principal topological reference is [6]. We will be using two particular topological spaces in our discussion; $\beta N - N$ (the remainder of the Stone-Čech compactification of the integers), and 2^{ω_1} with the usual product topology. $\beta N - N$ has weight c. 2^{ω_1} has weight ω_1.[1] We recall the following result, due to Parovičenko.

1.2. THEOREM [5]. *Every compact space having weight at most* ω_1 *is the continuous image of* $\beta N - N$.

Our version of $\diamond$ involves convergence. Recall that, if $(D, <)$ is a directed set, and if $(x_\alpha: \alpha \in D)$ is a net in the space X directed by D, and if $p \in X$,

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[1]Recall that the *weight* of a topological space X is the least cardinal of a basis for the topology of X.

then $(x_\alpha\colon \alpha \in D)$ *converges to* p if, for every neighborhood G of p in X there is $\alpha_0 \in D$ such that if $\alpha \in D$ and $\alpha \geqslant \alpha_0$, then $x_\alpha \in G$. The central concept in our theorem is the notion of a universal ω_1-sequence. Let X be a space and let $(x_\alpha\colon \alpha \in \omega_1)$ be an ω_1-sequence in X. We say $(x_\alpha\colon \alpha \in \omega_1)$ is *universal in* X if for every point $p \in X$ there is a stationary set $S \subseteq \omega_1$ so that the subnet $(x_\alpha\colon \alpha \in S)$ converges to p.

2. $\diamond$ **and universal ω_1-sequences.** We now give a version of $\diamond$ in topological terms.

2.1. THEOREM. *$\diamond$ is equivalent to the statement that any of the following spaces contains a universal ω_1-sequence*:

(i) $\beta N - N$,

(ii) 2^{ω_1},

(iii) *all spaces of weight* $\leqslant \omega_1$,

(iv) *all spaces of weight* $\leqslant c$.

PROOF. $\diamond \to$ (iv): Assume that $(S_\alpha\colon \alpha \in \omega_1)$ satisfies $\diamond$. As is well known, $\diamond \to$ CH. (Clearly every countable subset of ω_1 occurs among the S_α, and so $2^\omega = \omega_1$.) Now let X be a space of weight $\leqslant 2^\omega = \omega_1$; we can choose a basis $B = \{G_\alpha\colon \alpha < \omega_1\}$ for X indexed by ω_1. We define the sequence $(x_\alpha\colon \alpha \in \omega_1)$ as follows: if $\bigcap_{\beta \in S_\alpha} G_\beta \neq 0$, let x_α be any point in $\bigcap_{\beta \in S_\alpha} G_\beta$, and otherwise let x_α be an arbitrary point of X. We claim that $(x_\alpha\colon \alpha \in \omega_1)$ is universal in X. For, let $p \in X$. Let $A = \{\alpha \in \omega_1\colon p \in G_\alpha\}$. Then by $\diamond$, the set $S = \{\alpha \in \omega_1\colon A \cap \alpha = S_\alpha\}$ is stationary in ω_1. We now show that the subnet $(x_\alpha\colon \alpha \in S)$ converges to p. Let N be a neighborhood of p in X. Since B is a basis, there is an $\alpha_0 \in \omega_1$ such that $p \in G_{\alpha_0} \subseteq N$. Suppose $\alpha \in S$ and $\alpha > \alpha_0$. Then $S_\alpha = A \cap \alpha$, so $S_\alpha = \{\beta \in \alpha\colon p \in G_\beta\}$. Thus $\alpha_0 \in S_\alpha$, and $p \in \bigcap_{\beta \in S_\alpha} G_\beta$. Since $\bigcap_{\beta \in S_\alpha} G_\beta \neq 0$, we have $x_\alpha \in \bigcap_{\beta \in S_\alpha} G_\beta$. In particular, $x_\alpha \in G_{\alpha_0} \subseteq N$. Thus $\alpha \in S$ and $\alpha > \alpha_0$ implies $x_\alpha \in N$. Therefore $(x_\alpha\colon \alpha \in S)$ converges to p. This shows $(x_\alpha\colon \alpha \in \omega_1)$ is universal in X.

(iv) $\to$ (i): Trivial.

(i) $\to$ (ii): By 1.2, 2^{ω_1} is the continuous image of $\beta N - N$. Noting that a continuous map takes a universal sequence onto a universal sequence completes the proof of this implication.

(ii) $\to \diamond$: Let $(f_\alpha\colon \alpha \in \omega_1)$ be universal in 2^{ω_1}. We will show that for every $f \in 2^{\omega_1}$, $\{\alpha \in \omega_1\colon f|\alpha = f_\alpha|\alpha\}$ is stationary. There is a stationary set S in ω_1 so that $(f_\alpha\colon \alpha \in S)$ converges to f in 2^{ω_1}. Let $T = \{\alpha \in S\colon f_\alpha|\alpha \neq f|\alpha\}$. We claim that T is nonstationary. For, suppose T is stationary. For $\alpha \in T$, let x_α be the first ordinal at which f_α and f differ. Then $x_\alpha < \alpha$ for all $\alpha \in T$, and so the function $\alpha \to x_\alpha$ is pressing down on T. By 1.1, there is a stationary set $T_1 \subseteq T$ and an ordinal $\beta \in \omega_1$ so that $x_\alpha = \beta$ for all $\alpha \in T_1$. Thus $f_\alpha(\beta) \neq f(\beta)$ for all $\alpha \in T_1$. But $\{g \in 2^{\omega_1}\colon g(\beta) = f(\beta)\}$ is a neighborhood of f in 2^{ω_1}. Since $(f_\alpha\colon \alpha \in S)$ converges to f, this neighborhood contains all but countably many of the f_α, $\alpha \in S$. But T_1 is uncountable, and the neighborhood contains none of the f_α, $\alpha \in T$. This is a contradiction. Therefore T

is indeed nonstationary. Since S is stationary, $S - T$ is stationary, and so the set $\{\alpha \in \omega_1 : f_\alpha|\alpha = f|\alpha\}$, which contains $S - T$, is also stationary. We have shown that for every $f \in 2^{\omega}_1$, $\{\alpha \in \omega_1 : f|\alpha = f_\alpha|\alpha\}$ is stationary. Using the correspondence between a set and its characteristic function, it is easy to see that setting $S_\alpha = f_\alpha^{-1}\{1\} \cap \alpha$ gives a sequence $(S_\alpha : \alpha \in \omega_1)$ which satisfies $\diamond$.

Since, trivially, (iv) $\to$ (iii) $\to$ (ii), this completes the proof.

We remark that various other topological spaces can replace $\beta N - N$ or 2^{ω_1} in the preceding equivalence. Thus, for example, the existence of universal ω_1-sequences in βN or in 2^{ω_1} with the lexicographic order topology is equivalent to $\diamond$.

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