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A BEST POSSIBLE EXTENSION OF THE HAUSDORFF-YOUNG THEOREM

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ABSTRACT. The purpose of this note is to show that a recent result of A. M. Sedleckiĭ on nonharmonic Fourier series in $L^p(-\pi, \pi)$ has as a simple consequence a "best possible" extension of the classical Hausdorff-Young theorem.

THEOREM. *Let $1 < p \leqslant 2$ and let q be the conjugate exponent. Let $\{\lambda_n\}$ be any sequence of complex numbers for which*

$$\sup|\lambda_n - n| < (p-1)/2p \qquad (-\infty < n < \infty). \tag{1}$$

If $\{c_n\} \in l^p$, then there is a function f in $L^q(-\pi, \pi)$ such that

$$c_n = \hat{f}(\lambda_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{i\lambda_n t}\,dt. \tag{2}$$

If condition (1) *is replaced by the weaker condition*

$$|\lambda_n - n| < (p-1)/2p, \tag{3}$$

then the conclusion of the theorem is false.

PROOF. Under the conditions of the theorem, there is an isomorphism of $L^p(-\pi, \pi)$ which maps e^{int} into $e^{i\lambda_n t}$ [4]. It follows that for some constant $m > 0$ and every finite sequence $\{a_n\}$,

$$m\left\|\sum a_n e^{int}\right\|_p \leqslant \left\|\sum a_n e^{i\lambda_n t}\right\|_p.$$

This, together with the Hausdorff-Young theorem, shows that

$$m\left(\sum |a_n|^q\right)^{1/q} \leqslant \left\|\sum a_n e^{i\lambda_n t}\right\|_p.$$

But the above inequality guarantees [2] that the equations in (2) are solvable for f in $L^q(-\pi, \pi)$, and the first half of the theorem is established.

To see that condition (1) cannot be replaced by (3), let

$$\mu_n = \begin{cases} n - (p-1)/2p, & n > 0, \\ 0, & n = 0, \\ n + (p-1)/2p, & n < 0, \end{cases}$$

and choose $\{\lambda_n\}$ such that

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$$|\lambda_n - n| < (p-1)/2p \quad \text{and} \quad \sum |\lambda_n - \mu_n| < \infty.$$

Since $\{e^{i\mu_n t}\}_{n\neq 0}$ is complete in $L^p(-\pi, \pi)$ [3, p. 65], so too is $\{e^{i\lambda_n t}\}_{n\neq 0}$ [1, p. 66]. It follows, in particular, that there is no function f in $L^q(-\pi, \pi)$ for which $\hat{f}(\lambda_n) = 0$ $(n \neq 0)$ and $\hat{f}(\lambda_0) = 1$, and the proof is complete.

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