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σ-LOCALLY FINITE MAPS

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ABSTRACT. A map $f: X \to Y$ is called σ-locally finite if every σ-locally finite cover $\mathcal{A}$ of X has a refinement $\mathcal{B}$ such that $f(\mathcal{B})$ is σ-locally finite. The principal purpose of this paper is to provide proofs of some results on these maps which were announced by the author in a previous note.

1. **Introduction**. The concept of a σ-locally finite map was introduced in [4], primarily in order to characterize σ-spaces and Σ-spaces as indicated in Theorem 3.1 below. The main purpose of this note is to supply proofs of results which were only announced in [4]; the motivation for doing so at this time is that the characterization mentioned above is needed in the proof of a recent theorem of D.K. Burke and the author [2].

A map[1] $f: X \to Y$ is called *σ-locally finite* if every σ-locally finite cover[2] $\mathcal{A}$ of X has a refinement $\mathcal{B}$ such that $f(\mathcal{B})$ is σ-locally finite.[3] Our principal results about these maps are stated below. Regarding the terminology, recall that a space is *subparacompact* [1] if every open cover has a σ-locally finite closed refinement, and that a space is a *paracompact M-space* if it is Hausdorff and admits a perfect map onto a metric space; all other unfamiliar terms are defined in §3. In contrast to [4], where all spaces were assumed regular, we assume no separation properties unless specifically indicated.

PROPOSITION 1.1. *If $f: X \to Y$ is a map, then each of the following conditions implies that f is σ-locally finite.*

(a) *X is Lindelöf.*

(b) *f is perfect.*

(c) *f is closed, every $f^{-1}(y)$ is Lindelöf, and X or Y is subparacompact.*

(d) *X has an almost* (mod k)*-network $\mathcal{A}$ for which $f(\mathcal{A})$ is σ-locally finite.*

It was shown in [4, p. 6] that, in general, a closed map–even between paracompact spaces–need not be σ-locally finite.

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[1] Maps in this paper are continuous, but not necessarily onto.

[2] Covers need not be open covers.

[3] Here, and elsewhere in this paper, the phrase "$f(\mathcal{B})$ is σ-locally finite" is to be interpreted in the following strict, "indexed" sense: $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ so that, for all n, every $y \in Y$ has a neighborhood which intersects $f(B)$ for at most finitely many $B \in \mathcal{B}_n$. (This strict interpretation is required in the proof of Proposition 2.2(a) $\to$ (d).)

PROPOSITION 1.2. *Suppose* $f\colon X \to Y$ *is a* σ*-locally finite map onto a regular space* Y*. Then, if* X *has any of the following properties, so does* Y: (a) *subparacompact*, (b) σ*-space*, (c) *strong* Σ*-space.*

THEOREM 1.3. *The following properties of a regular space* Y *are equivalent.*

(a) Y *is a* σ*-space* (*resp. strong* Σ*-space*).

(b) Y *is the image under a* σ*-locally finite map* f *of a metrizable space* (*resp. paracompact* M*-space*).

Moreover, in (a) $\to$ (b) *for* σ*-spaces the map* f *can be chosen to be one-to-one, and in* (a) $\to$ (b) *for strong* Σ*-spaces the domain of* f *can be chosen to be a subset of* $Y \times M$ *for some metrizable space* M.

PROPOSITION 1.4. *If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are* σ*-locally finite maps, so is* $g \circ f\colon X \to Z$.

I would like to take this opportunity to point out an inaccuracy in one of the results of [4] (not involving σ-locally finite maps) which was kindly called to my attention by I. Juhasz: For Proposition 1 of [4] to be valid, one must assume that: (1) $\mathfrak{A}$ is preserved by finite intersections; (2) for each $x \in X$, $\bigcap\{A\colon x \in A \in \mathfrak{A}\} = \bigcap\{\bar{A}\colon x \in A \in \mathfrak{A}\}$. Alternatively, that result is valid if only assumption (1) is made, provided "(mod k)-network" is changed to "almost (mod k)-network". (See §3 below, particularly Proposition 3.2.)

In §2 we obtain some characterizations of σ-locally finite maps, and §3 proves some results related to Σ-spaces and (mod k)-networks which may be of independent interest. The results stated above are proved in §§4–7.

2. **σ-locally finite maps.** Before stating the main result of this section (Proposition 2.2), we need a definition and a lemma. If $\mathfrak{A}$ is a collection of subsets of X, then $\mathfrak{B}$ is a *base-like refinement* of $\mathfrak{A}$ if every $B \in \mathfrak{B}$ is a subset of some $A \in \mathfrak{A}$, and every $A \in \mathfrak{A}$ is the union of elements of $\mathfrak{B}$.

LEMMA 2.1. *Every locally finite collection* $\mathfrak{A}$ *of subsets of a space* X *has a disjoint, locally finite, base-like refinement* $\mathfrak{D}$ *such that every* $D \in \mathfrak{D}$ *intersects only finitely many* $A \in \mathfrak{A}$.

PROOF. For each finite $\mathfrak{F} \subset \mathfrak{A}$, let $D(\mathfrak{F}) = \bigcap \mathfrak{F} - \bigcup(\mathfrak{A} - \mathfrak{F})$. Let $\mathfrak{D}$ be the collection of all such $D(\mathfrak{F})$. It is easily checked that $\mathfrak{D}$ has all the required properties.

PROPOSITION 2.2. *If* $f\colon X \to Y$ *is a map, then* (a) $\to$ (b) $\leftrightarrow$ (c) $\leftrightarrow$ (d). *If* X *is subparacompact, then all four properties are equivalent.*

(a) *Every open cover* $\mathfrak{U}$ *of* X *has a refinement* $\mathfrak{B}$ *such that* $f(\mathfrak{B})$ *is* σ*-locally finite.*

(b) *Every* σ*-locally finite collection* $\mathfrak{A}$ *of subsets of* X *has a base-like refinement* $\mathfrak{B}$ *such that* $f(\mathfrak{B})$ *is* σ*-locally finite.*

(c) f *is* σ*-locally finite.*

(d) *Every locally finite cover* $\mathfrak{A}$ *of* X *has a refinement* $\mathfrak{B}$ *such that* $f(\mathfrak{B})$ *is* σ*-locally finite.*

PROOF. That (b) $\to$ (c) $\to$ (d) is obvious, and so is (c) $\to$ (a) if X is subparacompact. It remains to prove (a) $\to$ (d) $\to$ (b).

(a) $\to$ (d). Let $\mathfrak{A}$ be a locally finite cover of X. Let $\mathfrak{U}$ be an open cover of X such that each $U \in \mathfrak{U}$ intersects only finitely many $A \in \mathfrak{A}$. By (a), there is a refinement $\mathfrak{E}$ of $\mathfrak{U}$ such that $f(\mathfrak{E})$ is σ-locally finite. Let $\mathfrak{B} = \{A \cap E: A \in \mathfrak{A}, E \in \mathfrak{E}\}$. Then $\mathfrak{B}$ is a refinement of $\mathfrak{A}$, and it is easily checked that $f(\mathfrak{B})$ is σ-locally finite.

(d) $\to$ (b). It clearly suffices to prove (b) in case $\mathfrak{A}$ is locally finite. By Lemma 2.1, $\mathfrak{A}$ has a disjoint, locally finite, base-like refinement $\mathfrak{D}$. Let $\mathfrak{E} = \mathfrak{D} \cup \{X - \bigcup \mathfrak{D}\}$. Then $\mathfrak{E}$ is a locally finite cover of X, and hence has a refinement $\mathfrak{B}$ such that $f(\mathfrak{B})$ is σ-locally finite. Since $\mathfrak{E}$ is disjoint, $\mathfrak{B}$ must be a *base-like* refinement of $\mathfrak{E}$. Let $\mathfrak{B}' = \{B \in \mathfrak{B} : B \subset \bigcup \mathfrak{D}\}$. Then $\mathfrak{B}'$ is a base-like refinement of $\mathfrak{D}$ and thus of $\mathfrak{A}$, and $f(\mathfrak{B}')$ is σ-locally finite.

That completes the proof.

3. **Networks and spaces.** A cover $\mathfrak{A}$ of X is a *network* for X if, whenever $x \in U$ with U open in X, then $x \in A \subset U$ for some $A \in \mathfrak{A}$. A *σ-space* [8] is a space with a σ-locally finite closed network.[4] A cover $\mathfrak{A}$ of X is a (mod k)-*network* [4] for X if every $x \in X$ is in some compact $K_x \subset X$ such that, whenever $K_x \subset U$ with U open in X, then $K_x \subset A \subset U$ for some $A \in \mathfrak{A}$. A *strong Σ-space* [7] is a space with a σ-locally finite, closed (mod k)-network.

For some purposes (such as Lemma 5.1), it is convenient to consider a modification of (mod k)-networks, obtained by weakening $K_x \subset A \subset U$ to $x \in A \subset U$ in the above definition. We call this modification an *almost* (mod k)-*network*.[5] It will be shown (see Proposition 3.2) that the two concepts coincide under rather mild restrictions, and that they are therefore interchangeable in the above definition of a Σ-space (see Corollary 3.3).

We begin with a lemma which is somewhat more general than necessary. If $\mathfrak{A}$ is a collection of subsets of X, we denote $\{\bar{A}: A \in \mathfrak{A}\}$ by $\bar{\mathfrak{A}}$.

LEMMA 3.1. *The following properties of a filter base $\mathfrak{A}$ on a space X are equivalent.*

(a) *There is a compact $K \subset X$ such that, if $U \supset K$ and U is open in X, then $U \supset A$ for some $A \in \mathfrak{A}$.*

(b) *Same as* (a), *and also requiring that $\bigcap \mathfrak{A} \subset K \subset \bigcap \bar{\mathfrak{A}}$.*

PROOF. That (b) implies (a) is obvious. So let K be as in (a). Let

$$K' = (K \cup (\bigcap \mathfrak{A})) \cap (\bigcap \bar{\mathfrak{A}}).$$

We will show that K' satisfies the requirement of (b).

Since K is compact, and since every open set containing K contains $\bigcap \mathfrak{A}$, the set $K \cup (\bigcap \mathfrak{A})$ is also compact, and hence so is its closed subset K'.

[4] Some authors (in particular, A. Okuyama [8]) do not assume that the network is closed (i.e. consists of closed sets). If the space is regular, it makes no difference.

[5] I. Juhasz [3] calls this a *K-net*.

Now suppose $K' \subset U$, with U open in X, and let us show that $A \subset U$ for some $A \in \mathcal{A}$. Let $D = K - U$. Then D is compact, and $D \cap (\bigcap \bar{\mathcal{A}}) = \varnothing$, so $D \cap \bar{A}_1 = \varnothing$ for some $A_1 \in \mathcal{A}$. Let $V = X - \bar{A}_1$. Then $K \subset (U \cup V)$, and $U \cup V$ is open in X, so $A_2 \subset (U \cup V)$ for some $A_2 \in \mathcal{A}$. Pick $A \in \mathcal{A}$ with $A \subset (A_1 \cap A_2)$. Then $A \subset (U \cup V)$ and $A \cap V = \varnothing$, so $A \subset U$. That completes the proof.

PROPOSITION 3.2. *Suppose $\mathcal{A}$ is a cover of X which is preserved by finite intersections, such that $\bigcap(\mathcal{A}_x) = \bigcap(\bar{\mathcal{A}}_x)$ for all $x \in X$, where $\mathcal{A}_x$ denotes $\{A \in \mathcal{A}: x \in A\}$. Then $\mathcal{A}$ is a* (mod k)*-network for X if and only if $\mathcal{A}$ is an almost* (mod k)*-network for X.*

PROOF. The nontrivial part of this result follows immediately from Lemma 3.1, applied to $\mathcal{A}_x$.

The following corollary is needed in §5 to prove Proposition 1.2.

COROLLARY 3.3. *A space X is a strong Σ-space if and only if it has a σ-locally finite, closed almost* (mod k)*-network.*

PROOF. To prove the nontrivial half, let $\mathcal{A}$ be a σ-locally finite, closed almost (mod k)-network for X. We may suppose that $\mathcal{A}$ is preserved by finite intersections, so we can apply Proposition 3.2 to conclude that $\mathcal{A}$ is a (mod k)-network for X. Hence X is a strong Σ-space.

4. **Proof of Proposition 1.1.** Let us verify assertions (a)–(d).

(a) This is clear, since a σ-locally finite cover of a Lindelöf space is countable.

(b) This is true because the image of a locally finite collection under a perfect map is again locally finite.

(c) If X is subparacompact, then, since f is closed, $f(X)$ is also subparacompact by a result of D. K. Burke [1, Theorem 3.1]. We may therefore suppose that $f(X)$ is subparacompact.

By Proposition 2.2 (a) $\rightarrow$ (c), we need only show that every open cover $\mathcal{U}$ of X has a refinement $\mathcal{B}$ such that $f(\mathcal{B})$ is σ-locally finite. Our assumptions imply that there is a σ-locally finite cover $\mathcal{S}$ of $f(X)$ such that, for all $S \in \mathcal{S}$, $f^{-1}(S)$ is covered by countably many $U \in \mathcal{U}$, say $\{U_n(S): n \in N\}$. Let

$$\mathcal{B}_n = \{U_n(S) \cap f^{-1}(S): S \in \mathcal{S}\},$$

and let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. Then $\mathcal{B}$ is a refinement of $\mathcal{U}$, and $f(\mathcal{B})$ is σ-locally finite.

(d) Let $\mathcal{A}$ be an almost (mod k)-network for X such that $f(\mathcal{A})$ is σ-locally finite. We will verify that f satisfies 2.2(a). So let $\mathcal{U}$ be an open cover of X, and let $\mathcal{U}^*$ be the collection of finite unions of elements of $\mathcal{U}$. To show that $\mathcal{U}$ has a refinement $\mathcal{B}$ such that $f(\mathcal{B})$ is σ-locally finite, it suffices to show that $\mathcal{U}^*$ has such a refinement.

Since $\mathcal{A}$ is an almost (mod k)-network for X, and since $\mathcal{U}^*$ is closed under finite unions, $\mathfrak{U}^*$ has a refinement $\mathfrak{B}$ such that $\mathfrak{B} \subset \mathfrak{C}$. But then $f(\mathfrak{B})$ is σ-locally finite, and that completes the proof.

5. **Proof of Proposition 1.2.** We begin with two results which require no separation properties. The simple verification of Lemma 5.1 is omitted.

LEMMA 5.1. *Suppose that* $\mathfrak{C}$ *is a network* (*resp. almost* (mod k)-*network*) *for* X, *that* $\mathfrak{B}$ *is a base-like refinement of* $\mathfrak{C}$, *and that* f: $X \to Y$ *is an onto map. Then* $f(\mathfrak{B})$ *is a network* (*resp. almost* (mod k)-*network*) *for* Y.

PROPOSITION 5.2. *Suppose* f: $X \to Y$ *is a* σ-*locally finite map onto* Y. *Then, if* X *has any of the following properties, so does* Y.

(a) *Every open cover has a* σ-*locally finite refinement.*

(b) *There is a* σ-*locally finite network.*

(c) *There is a* σ-*locally finite almost* (mod k)-*network.*

PROOF. (a). Let $\mathcal{V}$ be an open cover of Y. Then $f^{-1}(\mathcal{V})$ is an open cover of X, and hence it has a refinement $\mathfrak{B}$ such that $f(\mathfrak{B})$ is σ-locally finite. But then $f(\mathfrak{B})$ is a σ-locally finite refinement of $\mathcal{V}$.

(b) and (c). Let $\mathfrak{C}$ be a σ-locally finite network (resp. almost (mod k)-network) for X. By Proposition 2.2(c) $\to$ (b), $\mathfrak{C}$ has a base-like refinement $\mathfrak{B}$ such that $f(\mathfrak{B})$ is σ-locally finite. By Lemma 5.1, $f(\mathfrak{B})$ must be a network (resp. almost (mod k)-network) for Y.

That completes the proof.

PROOF OF PROPOSITION 1.2. In a regular space, each of the conditions (a)–(c) in Proposition 5.2 is equivalent to the same condition strengthened to require the relevant cover to be a closed cover. Hence Proposition 1.2 follows from Proposition 5.2 and—for part (c)—Corollary 3.3.

6. **Proof of Theorem 1.3.** That (b) $\to$ (a) in Theorem 1.3 follows from Proposition 1.2 and the facts that every metrizable space is a σ-space and every paracompact M-space is a strong Σ-space. It remains to prove (a) $\to$ (b).

(a) $\to$ (b) for σ-spaces. (We only need Y to be T_1.) Let $\mathfrak{C}$ be a σ-locally finite closed network for Y; we may suppose that $\mathfrak{C}$ is preserved by finite intersections. Let X be the set Y, retopologized by taking $\mathfrak{C}$ to be a base, and let f: $X \to Y$ be the identity map. Then f is continuous because $\mathfrak{C}$ is a network for Y. Hence $\mathfrak{C}$ is also σ-locally finite in X. Since Y is T_1, so is X. Since each $A \in \mathfrak{C}$ is open and closed in X, the space X is regular. By the Nagata-Smirnov theorem, X is therefore metrizable. That f is σ-locally finite follows from Proposition 1.1(d).

(a) $\to$ (b) for strong Σ-spaces. (We only need Y to be Hausdorff.) Let $\mathcal{S}$ be a σ-locally finite, closed (mod k)-network for Y; we may suppose that $\mathcal{S}$ is preserved by finite intersections. Apply Theorem 2.6 of [5] (which is applicable by [5, Proposition 3.2(a)]) to obtain a metric space M and an $X \subset Y \times M$ such that, letting $f = \pi_1|X$ and $g = \pi_2|X$ (where π_1 and π_2 are the coordinate projections), we have:

(1) g is a perfect map.

(2) There is a base $\mathcal{B}$ for M such that $fg^{-1}(\mathcal{B}) = \mathcal{S}$.

Now since Y is Hausdorff, so is X, and hence X is a paracompact M-space by (1). Since $\mathcal{B}$ is a base for M, and since g is perfect, $g^{-1}(\mathcal{B})$ is a (mod k)-network for X, so f: $X \to Y$ is a σ-locally finite map by (2) and Proposition 1.1(d).

7. **Proof of Proposition 1.4.** Let $\mathcal{Q}$ be a σ-locally finite cover of X, and let us find a refinement $\mathcal{B}$ of $\mathcal{Q}$ such that $g(f(\mathcal{B}))$ is σ-locally finite. First, pick a refinement $\mathcal{C}$ of $\mathcal{Q}$ such that $f(\mathcal{C})$ is σ-locally finite. This implies that $\mathcal{C} = \bigcup_{n=1}^{\infty}\mathcal{C}_n$ so that each $f(\mathcal{C}_n)$ is locally finite and each element of $f(\mathcal{C}_n)$ is the image of only finitely many elements of $\mathcal{C}_n$. For every n, use Lemma 2.1 to choose a locally finite, base-like refinement $\mathcal{D}_n$ of $f(\mathcal{C}_n)$ such that each $D \in \mathcal{D}_n$ intersects at most finitely many elements of $f(\mathcal{C}_n)$. Then each $D \in \mathcal{D}_n$ intersects $f(C)$ for at most finitely many $C \in \mathcal{C}_n$.

By Proposition 2.2(c) $\to$ (b), each $\mathcal{D}_n$ has a base-like refinement $\mathcal{E}_n$ such that $g(\mathcal{E}_n)$ is σ-locally finite. Let

$$\mathcal{B}_n = \{C \cap f^{-1}(E): C \in \mathcal{C}_n, E \in \mathcal{E}_n\},$$

and let $\mathcal{B} = \bigcup_{n=1}^{\infty}\mathcal{B}_n$. Then $\mathcal{B}$ is a refinement of $\mathcal{C}$ and thus of $\mathcal{Q}$. Moreover, since every $E \in \mathcal{E}_n$ intersects $f(C)$ for at most finitely many $C \in \mathcal{C}_n$, it is easily checked that $g(f(\mathcal{B}))$ is σ-locally finite (in the strict sense required by footnote 3). That completes the proof.

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