A SHORT PROOF OF AN INEQUALITY OF CARLESON'S

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Abstract. We give a simple proof that if \( a_i, i = 1, 2, \ldots, \) is a uniformly separated sequence in the unit disk, then
\[
\sum (1 - |a_i|^2)|f(a_i)|^p < K\|f\|_p^p
\]
for all \( f \in H^p \) and \( 1 < p < \infty \).

Let \( \mathbb{C} \) denote the complex plane, \( \Delta(a,r) \) denote the disk \( \{ z: |z - a| < r \} \), and let \( H^p \) denote Hardy class \( H^p \) of \( \Delta(0,1) \) for \( 1 < p < \infty \). The key step in the duality proof of the Carleson interpolation theorem is to prove

Lemma 1. Let \( a_i, i = 1, 2, \ldots, \) be a uniformly separated sequence in \( \Delta(0,1) \). Then
\[
\sum (1 - |a_i|^2)|f(a_i)|^p < K\|f\|_p^p
\]
for all \( f \in H^p \) and \( 1 < p < \infty \).

If \( |B_i(a_i)| > \delta \) for all \( i \), where \( B_i \) is the Blaschke product vanishing on \( \{ a_j: j \neq i \} \), then the best value of \( K \) is \( C\ln(1/\delta) \), which comes from Carleson's proof [1] tempered with maximal functions. (I want to thank the referee for pointing this out to me.) We shall give a much simpler proof that \( K < \infty \), although our value for \( K \) will not be as good. We shall show that if \( 0 < \delta < \frac{1}{2} \), then we may choose
\[
K = 32/\delta^2.
\]

We shall need the Hilbert space \( A^{2,1} = \{ f \text{ analytic on } \Delta(0,1): \pi^{-1}f_{\Delta(0,1)}|f(z)|^2(1 - |z|^2) \ dx \ dy < \infty \} \). The \( A^{2,1} \) norm of \( f(z) = \Sigma c_n z^n \) is easily seen to be \( \Sigma |c_n|/(n + 1)(n + 2) \). Clearly \( A^{2,1} = \{ f': f \in H^2 \} \) and \( \| f' \|_{2,1} < \| f \|_2 \). The following lemma, concerning sums of squares of normalized point evaluations in \( A^{2,1} \), was given in slightly different form by Shapiro and Shields [4, p. 529].

Lemma 2. Let \( b_i, i = 1, 2, \ldots, \) be a sequence in \( \Delta(0,1) \). Suppose \( 0 < \eta < \frac{1}{2} \) and \( |b_i - b_j|/|1 - b_i b_j| > \eta \) if \( i \neq j \). Then
\[
\sum (1 - |b_i|^2)^3|f(b_i)|^2 < 32\eta^{-2}\|f\|_{2,1}^2
\]
for all \( f \in A^{2,1} \).

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PROOF. A little computation using the conformal invariance of the pseudo-
hyperbolic metric $\rho(b_i, b_j) = |b_i - b_j|/|1 - b_i b_j|$ shows that
$$\Delta(b_i, \left(\frac{1}{4}\right)\eta(1 - |b_i|^2)) \subseteq \{z: \rho(b_i, z_j) < \eta\}.$$ Thus the disks $\Delta(b_i, \left(\frac{1}{4}\right)\eta(1 - |b_i|^2)) = \Delta(b_i, s_i)$, $i = 1,2, \ldots$, are disjoint. Hence
\[
(1 - b_i^2)^3 |f(b_i)|^2 < 16\pi^{-1}\eta^{-2} \int_{\Delta(b_i,s_i)} (1 - |b_i|^2)|f(z)|^2 dx \, dy
\]
(2)
\[
< 32\pi^{-1}\eta^{-2} \int_{\Delta(b_i,s_i)} (1 - |z|^2)|f(z)|^2 dx \, dy.
\]
By summing (2) over all $i$, we obtain Lemma 2.

PROOF OF Lemma 1. It clearly suffices to prove Lemma 1 for $p = 2$ (cf. [2, p. 152]). Let $B$ be the Blaschke product vanishing at $\{a_i: i = 1,2, \ldots \}$. Let $f \in H^2$. Then $\|{(Bf)}^\prime\|_{2,1} < \|Bf\|_2 = \|f\|_2$. But
$$|(Bf)'(a_i)| = |B'(a_i) f(a_i)| = (1 - |a_i|^2)^{-1} |B_i(a_i)f(a_i)|.$$ Thus
$$\sum (1 - |a_i|^2)^2 |f(a_i)|^2 = \sum |B_i(a_i)|^{-2} (1 - |a_i|^2)^3 |(Bf)'(a_i)|^2
\[
< 32\delta^{-4}\|{(Bf)}^\prime\|_{2,1}^2 < 32\delta^{-4}\|f\|_2^2.
\]
This completes the proof of Lemma 1 and of equality (1).

REFERENCES


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