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A SHORT PROOF OF AN INEQUALITY OF CARLESON'S

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ABSTRACT. We give a simple proof that if a_i, $i = 1,2,\ldots$, is a uniformly separated sequence in the unit disk, then $\Sigma(1 - |a_i|^2)|f(a_i)|^p < K\|f\|_p^p$, for all $f \in H^p$ and $1 < p < \infty$.

Let $\mathbf{C}$ denote the complex plane, $\Delta(a,r)$ denote the disk $\{z: |z - a| < r\}$, and let H^p denote Hardy class H^p of $\Delta(0,1)$ for $1 \leqslant p \leqslant \infty$. The key step in the duality proof of the Carleson interpolation theorem is to prove

LEMMA 1. *Let a_i, $i = 1,2,\ldots$, be a uniformly separated sequence in $\Delta(0,1)$. Then*

$$\sum \left(1 - |a_i|^2\right)|f(a_i)|^p \leqslant K\|f\|_p^p$$

for all $f \in H^p$ and $1 \leqslant p < \infty$.

If $|B_i(a_i)| \geqslant \delta$ for all i, where B_i is the Blaschke product vanishing on $\{a_j: j \neq i\}$, then the best value of K is $C\ln(1/\delta)$, which comes from Carleson's proof [1] tempered with maximal functions. (I want to thank the referee for pointing this out to me.) We shall give a much simpler proof that $K < \infty$, although our value for K will not be as good. We shall show that if $0 < \delta \leqslant \frac{1}{2}$, then we may choose

$$K = 32/\delta^4. \tag{1}$$

We shall need the Hilbert space $A^{2,1} = \{f$ analytic on $\Delta(0,1)$: $\pi^{-1}\int_{\Delta(0,1)}|f(z)|^2(1 - |z|^2)\,dx\,dy < \infty\}$. The $A^{2,1}$ norm of $f(z) = \Sigma c_n z^n$ is easily seen to be $\Sigma|c_n|^2/(n + 1)(n + 2)$. Clearly $A^{2,1} = \{f': f \in H^2\}$ and $\|f'\|_{2,1} \leqslant \|f\|_2$. The following lemma, concerning sums of squares of normalized point evaluations in $A^{2,1}$, was given in slightly different form by Shapiro and Shields [4, p. 529].

LEMMA 2. *Let b_i, $i = 1,2,\ldots$, be a sequence in $\Delta(0,1)$. Suppose $0 < \eta \leqslant \frac{1}{2}$ and $|b_i - b_j|/|1 - b_i\bar{b}_j| \geqslant \eta$ if $i \neq j$. Then*

$$\sum \left(1 - |b_i|^2\right)^3 |f(b_i)|^2 \leqslant 32\eta^{-2}\|f\|_{2,1}^2$$

for all $f \in A^{2,1}$.

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PROOF. A little computation using the conformal invariance of the pseudo-hyperbolic metric $\rho(b_i,b_j) = |b_i - b_j|/|1 - b_i\bar{b}_j|$ shows that

$$\Delta\left(b_i, \left(\tfrac{1}{2}\right)\eta\left(1 - |b_i|^2\right)\right) \subseteq \left\{z: \rho(b_i, z_j) < \eta\right\}.$$

Thus the disks $\Delta(b_i, (\frac{1}{4})\eta(1 - |b_i|^2)) = \Delta(b_i,s_i)$, $i = 1,2, \ldots,$ are disjoint. Hence

$$\begin{aligned}(1 - b_i^2)^3|f(b_i)|^2 &\leqslant 16\pi^{-1}\eta^{-2}\int_{\Delta(b_i,s_i)}\left(1 - |b_i|^2\right)|f(z)|^2 dx\, dy \\ &\leqslant 32\pi^{-1}\eta^{-2}\int_{\Delta(b_i,s_i)}\left(1 - |z|^2\right)|f(z)|^2 dx\, dy.\end{aligned} \tag{2}$$

By summing (2) over all i, we obtain Lemma 2.

PROOF OF LEMMA 1. It clearly suffices to prove Lemma 1 for $p = 2$ (cf. [2, p. 152]). Let B be the Blaschke product vanishing at $\{a_i: i = 1,2, \ldots\}$. Let $f \in H^2$. Then $\|(Bf)'\|_{2,1} \leqslant \|Bf\|_2 = \|f\|_2$. But

$$|(Bf)'(a_i)| = |B'(a_i)f(a_i)| = \left(1 - |a_i|^2\right)^{-1}|B_i(a_i)f(a_i)|.$$

Thus

$$\begin{aligned}\sum\left(1 - |a_i|^2\right)|f(a_i)|^2 &= \sum |B_i(a_i)|^{-2}\left(1 - |a_i|^2\right)^3|(Bf)'(a_i)|^2 \\ &\leqslant 32\delta^{-4}\|(Bf)'\|_{2,1}^2 \leqslant 32\delta^{-4}\|f\|_2^2.\end{aligned}$$

This completes the proof of Lemma 1 and of equality (1).

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