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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATION[1]

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ABSTRACT. In this paper, the asymptotic properties of solutions of a certain fourth order differential equation are considered. Sufficient conditions for oscillation are also given.

Introduction. This paper is concerned with the solutions of the differential equation

$$(y''' + p(x)y)' + p(x)y' + f(y) = 0 \tag{1}$$

where $p(x)$ is a continuously differentiable function defined on $[0, \infty)$ satisfying $\int^{\infty} p(x)\,dx = \infty$. The function f: $(-\infty, \infty) \to (-\infty, \infty)$ is assumed to be continuous and satisfy the condition $f(y)/y \geqslant m > 0$ for $y \neq 0$. Under these assumptions, continuable nontrivial solutions of (1) with a multiple zero are oscillatory. (See Theorem 4.)

The motiviation for this study comes from a recent article by D. L. Lovelady [2]. In [2], Lovelady considers a special class of nonlinear fourth order equations and derives some oscillation criteria. We also refer to the works of J. W. Heidel [1] and P. Waltman [3] on nonlinear third order differential equations. Unlike the results in [1] and [3], *we do not require $p(x)$ to remain one-signed* in most of our results.

A solution $y(x)$ of (1) is said to be *continuable* if it exists on some ray $[a, \infty)$, $a > 0$. A nontrivial solution of (1) is *oscillatory* if it is continuable and has arbitrarily large zeros. By a *nonoscillatory* solution we mean a continuable solution which is not oscillatory. The term "solution" for the remainder of this work will mean a nontrivial continuable solution.

Main results. Our first result is essential to the results which follow.

LEMMA 1. *Let $y(x)$ be a solution of* (1). *Then*

$$F(y(x)) = y(x)[y'''(x) + p(x)y(x)] - y'(x)y''(x)$$

is nonincreasing on some ray $[a, \infty)$, in fact,

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$$\{F[y(x)]\}' = -(y''(x))^2 - y(x)f(y(x)).$$

Since $F[y(x)]$ is monotone it follows that $F[y(x)]$ is one-signed on some ray $[c, \infty)$. Using this fact, we will call a solution $u(x)$ of (1), *type* I, if $F[u(x)] \geqslant 0$ on some ray $[d, \infty)$. If $F[u(c)] < 0$ for some $c > 0$, then $u(x)$ is said to be *type* II.

Lemma 2. *Suppose $f(x)$ is a twice continuously differentiable function on $[a, \infty)$ satisfying $\int_a^\infty f''^2(x)\,dx < \infty$ and $\int_a^\infty f^2(x)\,dx < \infty$. Then*

$$\left(\int_a^\infty f'^2(x)\,dx\right)^2 < \int_a^\infty |f|^2 \cdot \int_a^\infty |f''|^2.$$

Proof. Expand f in a Fourier cosine series on $[a, a+T]$, use Parseval's equality and the C-B-S inequality, then let $T \to \infty$.

We now examine properties of type I solutions.

Theorem 3. *Let $y(x)$ be a type* I *solution. Then the following are true*:
(i) $\int^\infty y''^2(x)\,dx < \infty$ *and* $\int^\infty y(x)f(y(x))\,dx < \infty$,
(ii) $\int^\infty y^2(x)\,dx < \infty$,
(iii) $\int^\infty y'^2(x)\,dx < \infty$.

Proof. Since $y(x)$ is type I, $F[y(x)] \geqslant 0$ on $[a, \infty)$ for some $a > 0$. By differentiating $F[y(x)]$ and integrating from a to x we obtain

$$0 \leqslant F[y(x)] = F[y(a)] - \int_a^x y''^2(t)\,dt - \int_a^x y(t)f(y(t))\,dt.$$

This proves (i).

To prove (ii), note that $y(x)f(y(x)) \geqslant my^2(x)$, and apply (i).

Finally, the proof of (iii) follows immediately from (i), (ii) and Lemma 2.

We now consider the type II solutions.

Theorem 4. *Let $y(x)$ be a type* II *solution. Then $y(x)$ is oscillatory.*

Proof. Suppose $y(x)$ is eventually positive, then there exists $x = c$ such that $y(x) > 0$ on $[c, \infty)$ and $F[y(c)] < 0$.

Consider the function

$$J(x) = \frac{y''(x)}{y(x)} + \int_c^x p(t)\,dt.$$

By differentiating $J(x)$ we find

$$J'(x) = F[y(x)]/y^2(x)$$

for $x > c$. So $J(x)$ is decreasing on $[c, \infty)$. Since $\int_c^\infty p(x)\,dx = \infty$, it follows that $y''(x) < 0$ for large x. Since $y(x) > 0$ we must have $y'(x) > 0$ on some ray $[d, \infty)$, $d > c$. The fact that $\int_c^x p(t)\,dt \to \infty$ as $x \to \infty$ and $J(x)$ is decreasing implies $y''(x)/y(x) \to -\infty$ as $x \to \infty$. But this implies $y''(x)$ is bounded away from zero for large x, implying $y(x) \to -\infty$ as $x \to \infty$. This contradiction proves the theorem.

COROLLARY. *Any nontrivial solution of* (1) *with a multiple zero is oscillatory.*

LEMMA 5. *Suppose* $p(x) > 0$ *and let* $y(x)$ *be a type* II *solution. Then*

$$N[y(x)] = y(x)y''(x) - y'^2(x) \to -\infty$$

as $x \to \infty$.

PROOF. Note that $N'[y(x)] = F[y(x)] - p(x)y^2(x) < F[y(x)]$. Since $y(x)$ is a type II solution, $F[y(x)]$ is negative and bounded away from zero on some ray $[a, \infty)$ and the result follows.

As our final theorem we list some properties of type II solutions.

THEOREM 6. *Let* $y(x)$ *be a type* II *solution and assume* $p(x) > 0$. *Then*
(i) $y'(x)$ *is unbounded, and*
(ii) $\int_a^\infty y'^2(x)\,dx = \infty$.

PROOF. From Lemma 5, $N[y(x)] \to -\infty$ as $x \to \infty$. Since $y(x)$ is oscillatory (Theorem 4) (i) follows immediately by evaluating $N[y(x)]$ along the zeros of $y(x)$.

To prove (ii), integrate $N[y(x)]$ from c to x where $y'(c) = 0$. Doing so, we obtain

$$\int_c^x N[y(t)]\,dt = y(x)y'(x) - 2\int_c^x y'^2(t)\,dt.$$

But $\int_c^x N[y(t)]\,dt \to -\infty$ as $x \to \infty$ and, since $y(x)$ is oscillatory, (ii) follows.

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