PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 65, Number 1, July 1977

ON ANALYTICITY OF LOCAL RESOLVENTS AND EXISTENCE OF SPECTRAL SUBSPACES

CHE-KOA FONG

ABSTRACT. We present some sufficient conditions for a function from an open set in $\mathbf{C}$ into a Hilbert space H such that $(T - z)f(z) = x$ ($T \in B(H)$ and $x \in H$) to be analytic. As an application we show that hyperinvariant subspaces exist for certain class of operators.

Let T be a bounded operator on a Hilbert space H. Suppose f is a vector-valued mapping from an open set U in the complex plane $\mathbf{C}$ into H, y is a vector in H, and $(T - z)f(z) = y$ for all z in U. We ask what additional conditions force f to be analytic. For example, a recent work of Stampfli and Wadhwa [6] showed that if T is dominant and f is bounded, then f is analytic. (Also see [5].) In this note, we present some circumstances under which f is analytic. As an application we give a sufficient condition for the existence of hyperinvariant subspaces.

For a Hilbert space H, we shall write $B(H)$ for the set of all bounded operators on H. Let $T \in B(H)$ and F be a compact set in $\mathbf{C}$. We shall write $X_T(F)$ for the linear manifold consisting of those x in H such that $(T - z)f(z) \equiv x$ for some analytic vector-valued function f from $\mathbf{C}\backslash F$ into H. For convenience, we call the closure of $X_T(F)$ a *spectral subspace* of T. Obviously, a spectral subspace of T is always hyperinvariant for T; that is, it is invariant for every operator commuting with T. For basic properties of spectral manifolds $X_T(F)$ we refer to [1]. We shall write $\operatorname{Sp}(T)$ for the spectrum of T and $\Pi(T)$ for the approximate point spectrum of T. For the definition and basic properties of approximate point spectra, see Chapter 8 in [2]. For $F \subseteq \mathbf{C}$, we write F^* for $\{\bar{z}: z \in F\}$.

PROPOSITION 1. *If $T \in B(H)$ and $y \in \bigcap_{z \in U}(T - z)H$ where U is an open set in $\mathbf{C}$ such that $U \cap \Pi(T) = \varnothing$, then $z \to (T - z)^{-1}y$ is an analytic vector-valued function.*

PROOF. For convenience, write $f(z) = (T - z)^{-1}y$ ($z \in U$). (This function is uniquely defined, by the hypothesis on y.) First we show that f is bounded on compacta. If not, there exists a convergent sequence $\{z_n\}$ in U such that $z_0 = \lim_n z_n \in U$ and $\lim_n \|f(z_n)\| = \infty$. Let $x_n = \|f(z_n)\|^{-1}f(z_n)$. Then

Received by the editors September 15, 1976 and, in revised form, October 8, 1976.
AMS (MOS) subject classifications (1970). Primary 47A10, 47A15; Secondary 47B20.
Key words and phrases. Approximate point spectrum, spectral subspace, M-hyponormal operator.

$\|x_n\| = 1$ and

$$\|(T - z_0)x_n\| \leqslant \|(T - z_n)x_n\| + \|(z_n - z_0)x_n\|$$
$$= \|f(z_n)\|^{-1}\|y\| + |z_n - z_0| \to 0$$

as $n \to \infty$. This contradicts the fact that $z_0 \notin \Pi(T)$.

Let $z_0 \in U$. Then, for $z \neq z_0$, we have

$$(*) \qquad (T - z_0)g(z) = f(z),$$

where

$$(**) \qquad g(z) = (z - z_0)^{-1}(f(z) - f(z_0)).$$

Since $z_0 \notin \Pi(T)$, the operator $T - z_0$ is bounded below and thus has a bounded inverse when it is considered as a linear map from H onto its range $(T - z_0)H$ (which is closed). (We shall designate this inverse by $(T - z_0)^{-1}$.) From $(*)$ we see that g is bounded on a neighborhood of z_0 and hence, by $(**)$, f is continuous at z_0. We have shown that f is a continuous function. By $(*)$ again, we have

$$\lim_{z \to z_0} g(z) = \lim_{z \to z_0} (T - z_0)^{-1} f(z) = (T - z_0)^{-1} f(z_0).$$

Hence, by $(**)$, f is differentiable at z_0. The proof is complete.

REMARK. Since the boundary of $\mathrm{Sp}(T)$ is contained in $\Pi(T)$ and is a closed set, the open set U in the above proposition is a disjoint union of two open subsets U_1 and U_2 with $U_1 \subset \mathrm{Sp}(T)$ and $U_2 \cap \mathrm{Sp}(T) = \varnothing$. It is well known that the map $z \mapsto (T - z)^{-1}$ is analytic on U_2. Hence our interest of the proposition is the case when $U \subset \mathrm{Sp}(T)\backslash\Pi(T)$.

COROLLARY. *If $T \in B(H)$, F is a compact set in $\mathbf{C}$ and $R \supseteq \Pi(T)$, then $X_T(F)$ is closed.*

PROOF. In fact, by Proposition 1, we have

$$X_T(F) = \bigcap_{z \in \mathbf{C}\backslash F} (T - z)H$$

where each $(T - z)H$ is closed.

PROPOSITION 2. *Let $T \in B(H)$, F be a compact set in $\mathbf{C}$ and $x \in H$. If $f\colon \mathbf{C}\backslash F \to H$ is a bounded vector-valued function such that $(T - z)f(z) \equiv x$ and $X_{T^*}(F^*)$ is dense in H, then f is analytic.*

PROOF. Since f is bounded, it suffices to show that the map $z \mapsto (f(z), y)$ is analytic for each y in a dense subset of H. Let $y \in X_{T^*}(F^*)$. Then there is an analytic function $g\colon \mathbf{C}\backslash F^* \to H$ such that $(T^* - \bar{z})g(\bar{z}) = y$ for $z \notin F$. Hence, for $z \notin F$, we have

$$(f(z),y) = (f(z),(T^* - \bar{z})g(\bar{z})) = ((T - z)f(z),g(\bar{z})) = (x,g(\bar{z})).$$

Clearly $z \mapsto (x,g(\bar{z}))$ is analytic. The proof is complete.

REMARK. The above results can be easily generalized to operators on Banach spaces.

As an application of Proposition 2, we have the following:

PROPOSITION 3. *Let $T \in B(H)$. Suppose: (1) there is a nonzero invariant subspace K of T such that $T|K$ is a normal operator, and (2) there is a nonzero positive operator P such that $(T - z)^*(T - z) \geqslant P^2$ for all $z \in \mathbf{C}$. Then T has a nontrivial spectral subspace.*

PROOF. Let $y \in H$ be a vector such that $x = Py \neq 0$. By Putnam [3, Theorem 6], there exists a bounded vector-valued function f: $\mathbf{C} \to H$ such that $(T^* - z)f(z) = x$.

Let E be the resolution of identity for $T|K$ and $\mathfrak{D}$ be the collection of all closed discs D in $\mathbf{C}$ such that $\mathrm{Sp}(T|K) \cap$ (interior of D) $\neq \varnothing$. For $D \in \mathfrak{D}$, we have $X_T(D) \supseteq E(D)K \neq \{0\}$. Hence it suffices to show that $X_T(D)$ is not dense in H for some D in $\mathfrak{D}$. Suppose otherwise. Then, by Proposition 2, for each $D \in \mathfrak{D}$, f is analytic on $\mathbf{C}\backslash D^*$. Hence f is a bounded entire function with

$$\lim_{|z|\to\infty} f(z) = \lim_{|z|\to\infty} (T^* - z)^{-1}x = 0.$$

By Liouville's theorem, $f = 0$, contradicting $x \neq 0$ and $(T^* - z)f(z) = x$. The proof is complete.

COROLLARY. *Let $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$. Suppose: (1) there is a nontrivial invariant subspace K of T_1 such that $T_1|K$ is a normal operator, and (2) T_2 is a nonscalar M-hyponormal operator (that is,*

$$(T_2 - z)(T_2 - z)^* \leqslant M(T_2 - z)^*(T_2 - z)$$

for all z in $\mathbf{C}$). Then $T = T_1 \oplus T_2$ has a nontrivial spectral subspace.

PROOF. Let P be the positive square root of $(T_2^* T_2 - T_2 T_2^*)^2$. If $P = 0$, then T_2 is normal and we are done. Hence we may assume that $P \neq 0$. By Radjabalipour [5, Theorem 2], there is a positive number k such that $(T - z)^*(T - z) \geqslant kP^2$ for all z in $\mathbf{C}$. Now the corollary follows from Proposition 3.

REMARK. We do not know in the above corollary if T_1 and T_2 separately have nontrivial hyperinvariant subspaces (provided they are nonscalar operators).

ACKNOWLEDGEMENT. The author would like to thank M. D. Choi for his help which leads to a simplified proof of Proposition 1.

REFERENCES

1. I. Colojoară and C. Foiaş, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.

2. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967. MR **34** #8178.

3. C. R. Putnam, *Hyponormal contractions and strong power convergence*, Pacific J. Math. **57** (1975), no. 2, 531–538. MR **52** #1393.

4. M. Radjabalipour, *Ranges of hyponormal operators* (to appear).

5. ———, *On majorization and normality of operators*, Proc. Amer. Math. Soc. **62** (1977), 105–110.

6. J. G. Stampfli and B. L. Wadhwa, *On dominant operators* (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO M5S 1A1, CANADA