PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 65, Number 1, July 1977

THE DENSITY CHARACTER OF UNIONS

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ABSTRACT. We consider only completely regular, Hausdorff spaces. Responding to a question of R. Levy and R. H. McDowell [Proc. Amer. Math. Soc. 49 (1975), 426–430] we show that for $\omega < \gamma < 2^{2^\omega}$ there is a separable space equal to the (appropriately topologized) disjoint union of γ copies of the "Stone-Čech remainder" $\beta N \setminus N$. More generally, denoting density character by d and weight by w, we prove this

THEOREM. *The following statements about infinite cardinal numbers* γ *and* α *are equivalent*: (a) $2^\alpha < 2^\gamma$ *and* $\gamma < 2^{2^\alpha}$; (b) *For every family* $\{X_\xi: \xi < \gamma\}$ *of spaces, with* $w(X_\xi) \leq 2^\alpha$ *for all* $\xi < \gamma$, *the set-theoretic disjoint union* $X = \bigcup_{\xi<\gamma} X_\xi$ *admits a topology such that* $d(X) \leq \alpha$ *and each* X_ξ *is a topological subspace of* X.

The following observation (a special case of Theorem 3.1) suggests that it may be difficult to achieve a stronger result: If $\alpha > \omega$ and X_0 and X_1 denote copies of the discrete space of cardinality α^+, then the disjoint union $X = X_0 \cup X_1$ admits a topology (making each X_i a topological subspace) such that $d(X) \leq \alpha$.

1. **Notation and references to the literature.** By a "space" we mean a completely regular, Hausdorff space. The symbols d and w were defined in the abstract. For $\alpha \geq \omega$ we set

$$\log \alpha = \min\{\gamma: 2^\gamma \geq \alpha\}.$$

When $\alpha \geq \omega$ we denote also by the symbol α the discrete space of cardinality α, and by $\beta(\alpha)$ its Stone-Čech compactification. As usual we identify $\beta(\alpha)$ with the set of ultrafilters on α, topologized so that

$$\{\{p \in \beta(\alpha): A \in p\}: A \subset \alpha\}$$

is a base for the closed sets; evidently $w(\beta(\alpha)) \leq 2^\alpha$, so that $w(X) \leq 2^\alpha$ for all $X \subset \beta(\alpha)$. We set

$$U(\alpha) = \{p \in \beta(\alpha): |A| = \alpha \text{ for all } A \in p\},$$

and we recall (see for example Corollary 7.15 of [1]) that there are $p \in U(\alpha)$ with no basis of cardinality $< 2^\alpha$. Thus we have:

1.1. If $\alpha \geq \omega$, then $w(U(\alpha)) = 2^\alpha$.

Received by the editors April 12, 1976 and, in revised form, October 29, 1976.
AMS (MOS) subject classifications (1970). Primary 54A25; Secondary 54A10.
Key words and phrases. Density character, weight, Tychonoff space.

[1]This author gratefully acknowledges support received from the National Science Foundation under grant NSF-MPS73-08937A02.

It is well known and easy to prove that for every space X the family of regular-open subsets of X (i.e., the family of subsets U of X such that $U = \operatorname{int} \operatorname{cl} U$) is a base for X. Further, if D is dense in X and U and V are different regular-open subsets of X, then $U \cap D \neq V \cap D$. This proves 1.2 below. Statements 1.3, 1.4 and 1.5 are equally familiar. For proofs, see for example Corollaries 2.11, 3.18, and 12.20 (together with Lemma 7.12 (b)) of [1].

1.2. If X is a space then $w(X) \leqslant 2^{d(X)}$.

We denote the real line in its usual topology by the symbol $\mathbf{R}$.

1.3. If X is a space such that $w(X) \leqslant \alpha$, then X is (homeomorphic with) a subspace of $\mathbf{R}^\alpha$; thus $|X| \leqslant 2^{w(X)}$.

1.4. If $\alpha \geqslant \omega$ then $d(\mathbf{R}^{2^\alpha}) \leqslant \alpha$.

1.5. If $\alpha \geqslant \omega$ and V is a nonempty, open subset of $U(\alpha)$, then there is a family $\mathcal{U}$ of pairwise disjoint, nonempty open subsets of V such that $|\mathcal{U}| = \alpha^+$; hence $d(V) > \alpha$.

2. **Topologizing a disjoint union.** If X_ξ is a subspace of a space X such that $d(X) \leqslant \alpha$, then from 1.2 above we have $w(X_\xi) \leqslant w(X) \leqslant 2^\alpha$. This explains the presence of the hypothesis "$w(X_\xi) \leqslant 2^\alpha$" in the following result.

2.1. THEOREM. *Let α and γ be cardinals, with $\alpha > \omega$. The following statements are equivalent.*

(a) $\log 2^\alpha \leqslant \gamma \leqslant 2^{2^\alpha}$;

(b) *for every family $\{X_\xi: \xi < \gamma\}$ of (pairwise disjoint) nonempty spaces, with $w(X_\xi) \leqslant 2^\alpha$ for all $\xi < \gamma$, the set-theoretic disjoint union $X = \bigcup_{\xi<\gamma} X_\xi$ admits a topology such that $d(X) \leqslant \alpha$ and each X_ξ is a topological subspace of X.*

PROOF. (a) $\Rightarrow$ (b). Let $w(X_\xi) \leqslant 2^\alpha$ for all $\xi < \gamma$, define $\delta = \log 2^\alpha$, using 1.4 above let $D = \{p(\xi): \xi < \delta\}$ be a faithfully indexed dense subset of $\mathbf{R}^{2^\alpha}$, and choose $p(\delta) \in \mathbf{R}^{2^\alpha} \setminus D$. For $S \subset 2^\alpha$ we denote by π_S the projection from $\mathbf{R}^{2^\alpha}$ onto $\mathbf{R}^S$, and we choose $A \subset 2^\alpha$ such that $|A| = \delta$ and $\pi_A | D \cup \{p(\delta)\}$ is a one-to-one function. For $\xi \leqslant \delta$ we define

$$G_\xi = \pi_A^{-1}\left(\pi_A(p(\xi))\right) = \left\{x \in \mathbf{R}^{2^\alpha}: x_\eta = p(\xi)_\eta \text{ for all } \eta \in A\right\},$$

and we note (since $\delta \leqslant \alpha < 2^\alpha$) that G_ξ is homeomorphic to $\mathbf{R}^{2^\alpha}$. It follows from 1.3 above that for $\xi < \delta$ the space X_ξ is (homeomorphic with) a subspace of G_ξ; we assume without loss of generality, using the fact that G_ξ is a homogeneous space, that $p(\xi) \in X_\xi$ for all $\xi < \delta$.

If $\gamma = \delta$ then since D is dense in $\mathbf{R}^{2^\alpha}$ and

$$D \subset X = \bigcup_{\xi<\gamma} X_\xi \subset \bigcup_{\xi<\gamma} G_\xi \subset \mathbf{R}^{2^\alpha}$$

we have $d(X) \leqslant \alpha$ and the proof is complete. If $\delta < \gamma \leqslant 2^{2^\alpha}$ then we note that since G_δ is homeomorphic with $\mathbf{R}^{2^\alpha}$, hence with $\mathbf{R}^{2^\alpha} \times \mathbf{R}^{2^\alpha}$, the space G_δ contains γ disjoint copies (indeed, 2^{2^α} disjoint copies) of $\mathbf{R}^{2^\alpha}$. Thus the spaces X_ξ (with $\delta \leqslant \xi < \gamma$) are homeomorphic with pairwise disjoint subspaces of G_δ

and again, giving $X = \bigcup_{\xi<\delta} X_\xi$ the topology inherited from $\mathbf{R}^{2^\alpha}$, we have $d(X) \leqslant \alpha$ because $D \subset X \subset \mathbf{R}^{2^\alpha}$ and D is dense in $\mathbf{R}^{2^\alpha}$.

(b) $\Rightarrow$ (a). From 1.2 and 1.3 we have $|X| \leqslant 2^{2^{d(X)}}$, so that necessarily $\gamma \leqslant 2^{2^\alpha}$.

Let $\gamma < \log 2^\alpha$, let $X_0 = U(\alpha)$ and for $0 < \xi < \gamma$ let X_ξ be the singleton space $\{\xi\}$, suppose that the set-theoretic disjoint union $X = \bigcup_{\xi<\gamma} X_\xi$ is topologized as in (b), and let D be a dense subset of X such that $|D| \leqslant \alpha$. If $D \setminus X_0$ is dense in X then we have $d(X) \leqslant \gamma$ and hence (from 1.1 and 1.2),

$$2^\alpha = w(X_0) \leqslant w(X) \leqslant 2^{d(X)} \leqslant 2^\gamma < 2^\alpha,$$

a contradiction. Thus there is a nonempty, open subset V of X_0 such that $V \subset \mathrm{cl}_{X_0}(D \cap X_0)$, so that $V \subset \mathrm{cl}_V(D \cap V)$. But then $d(V) \leqslant |D| \leqslant \alpha$, contrary to 1.5 above.

The proof is complete.

The following consequence of Theorem 2.1 was proved by R. Levy and R. H. McDowell [3] in the case $\omega \leqslant \gamma \leqslant 2^\omega$; they asked, in effect, if the result could be achieved for $2^\omega < \gamma \leqslant 2^{2^\omega}$. We note that in our abstract [2] we have outlined a proof of Corollary 2.2 based on the Levy-McDowell method of [3]; this method is quite different from those of the present paper.

2.2. Corollary. *If $\omega \leqslant \gamma \leqslant 2^{2^\omega}$, there is a separable space equal to the (appropriately topologized) disjoint union of γ copies of the space $U(\omega)$.*

3. **A final remark.** It is tempting to believe that for every collection $\{X_\xi: \xi < \gamma\}$ of spaces such that $\gamma < \log 2^\alpha$ and $d(X_\xi) > \alpha$ for all $\xi < \gamma$, the disjoint union $X = \bigcup_{\xi<\gamma} X_\xi$ admits no topology such that $d(X) \leqslant \alpha$ and each X_ξ is a topological subspace. The following simple example, though susceptible to substantial generalization, is sufficient to dispel this belief. Additional examples are expected in [4].

3.1. Theorem. *Let α and γ be cardinals with $\alpha \geqslant \omega$ and with $2 \leqslant \gamma \leqslant 2^{2^\alpha}$, and for $\xi < \gamma$ let X_ξ be a discrete space such that $|X_\xi| = \alpha^+$. Then the set-theoretic disjoint union $X = \bigcup_{\xi<\gamma} X_\xi$ admits a topology such that $d(X) = \alpha$ and each X_ξ is a topological subspace of X.*

Proof. Since $w(X_\xi) = \alpha^+ \leqslant 2^\alpha$, the case $\log 2^\alpha \leqslant \gamma$ is handled by Theorem 2.1. We assume in what follows that $\gamma \leqslant \alpha$.

Let f be a fixed-point-free permutation of γ, for $\xi < \gamma$ choose $D_\xi \subset X_\xi$ such that $|D_\xi| = \alpha$, and identify $X_{f(\xi)} \setminus D_{f(\xi)}$ with a (discrete) family of uniform ultrafilters over the discrete space D_ξ. (Such a family exists by 1.5 above.) Writing

$$Y_\xi = D_\xi \cup (X_{f(\xi)} \setminus D_{f(\xi)})$$

we have the topological inclusion $D_\xi \subset Y_\xi \subset \beta(D_\xi)$, so that $d(Y_\xi) = \alpha$. Now let X be the topological disjoint union of the spaces Y_ξ–i.e., a subset S of X is open if and only if $S \cap Y_\xi$ is open in Y_ξ for each $\xi < \gamma$. It is clear that $d(X) = \alpha$. Finally for $\xi < \gamma$ there is $\eta < \gamma$ such that $\eta \neq \xi$ and $\xi = f(\eta)$;

since D_ξ and $X_\xi \setminus D_\xi$ are disjoint discrete subsets of the disjoint open-and-closed subspaces Y_ξ and Y_η respectively, the set X_ξ is discrete in X, as required.

References

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