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THE TURÁN-KUBILIUS INEQUALITY

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ABSTRACT. The author gives a short proof of the Turán-Kubilius inequality with an improved constant.

Let $f(n)$ be a complex-valued strongly additive arithmetic function. Define

$$E(m) = \sum_{p \leqslant m} f(p)p^{-1}, \qquad B^2(m) = \sum_{p \leqslant m} |f(p)|^2 p^{-1}$$

where p runs through the prime numbers.

Let λ_m denote the smallest number for which the inequality

$$\sum_{n=1}^{m} |f(n) - E(m)|^2 \leqslant m\lambda_m B^2(m)$$

is always satisfied, and set

$$\lambda = \limsup_{m\to\infty} \lambda_m.$$

According to the Turán-Kubilius inequality (see Kubilius [2]) we have

$$1.47 \leqslant \lambda \leqslant 2.08.$$

The exact value of λ is not known.

We shall show that

THEOREM. $\lambda \leqslant 2$.

The proof is short. Let μ_m denote the number which takes the place of λ_m when $f(m)$ is restricted to real and nonnegative values. The desired result will follow from the two simpler results:

(i) $\lambda_m \leqslant 2\mu_m$,

(ii) $\limsup_{m\to\infty} \mu_m \leqslant 1$.

PROOF OF (i). Assume that $f(m)$ is real. Define new strongly additive functions by

$$g_1(p) = \begin{cases} f(p) & \text{if } f(p) \geqslant 0, \\ 0 & \text{if } f(p) < 0; \end{cases}$$

$$g_2(p) = \begin{cases} -f(p) & \text{if } f(p) < 0, \\ 0 & \text{if } f(p) \geqslant 0. \end{cases}$$

Corresponding to these, define

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$$E_j = \sum_{p \leqslant m} g_j(p)p^{-1} \qquad (j = 1, 2).$$

Then, applying the Cauchy-Schwarz inequality to each summand:

$$\sum_{n=1}^{m} |f(n) - E|^2 \leqslant 2 \sum_{j=1}^{2} \sum_{n=1}^{m} |g_j(n) - E_j|^2$$

$$\leqslant 2 \sum_{j=1}^{2} m\mu_m \sum_{p \leqslant m} |g_j(p)|^2 p^{-1} = 2m\mu_m B^2(m).$$

This establishes (i) when $f(n)$ is real. If $f(n)$ is complex we define

$$g_1(n) = \operatorname{Re} f(n), \qquad g_2(n) = \operatorname{Im} f(n),$$

and (in an obvious abuse of notation):

$$\sum_{n=1}^{m} |f(n) - E|^2 = \sum_{j} \sum_{n=1}^{m} |g_j(n) - E_j|^2$$

$$\leqslant 2m\mu_m \sum_{p \leqslant m} p^{-1}\left(|g_1(p)|^2 + |g_2(p)|^2\right) = 2m\mu_m B^2(m).$$

Proof of (ii). A result of this type is already known (see Kubilius [2, p. 374]). We follow the original method of Turán [4] (and see also Kubilius [3]). The sum which we wish to estimate is

$$S = \sum_{n=1}^{m} f(n)^2 - 2E \sum_{n=1}^{m} f(n) + mE^2.$$

Inverting the order of summation, the first sum is

$$\sum_{p \leqslant m} f^2(p)\left[\frac{m}{p}\right] + \sum_{\substack{p \neq q \\ pq \leqslant m}} f(p)f(q)\left[\frac{m}{pq}\right] \leqslant mB^2(m) + mE^2.$$

The second sum is

$$-2E \sum_{p \leqslant m} f(p)\left[\frac{m}{p}\right] \leqslant -2mE^2 + 2E \sum_{p \leqslant m} f(p),$$

since $[y] > y - 1$ whenever $y \geqslant 0$. Collecting terms yields

$$S - mB^2(m) \leqslant 2E \sum_{p \leqslant m} f(p)p^{-1/2}p^{-1/2}$$

$$\leqslant \left(2B^2(m) \sum_{p \leqslant m} \frac{1}{p}\right)^{1/2} \left(B^2(m) \sum_{p \leqslant m} p\right)^{1/2},$$

this last step involving two appeals to the Cauchy-Schwarz inequality. Since

$$\sum_{p \leqslant m} \frac{1}{p} = O(\log \log m), \qquad \sum_{p \leqslant m} p = O\left(\frac{m^2}{\log m}\right)$$

(see, for example, Hardy and Wright [1, Theorems 427,7]), we see that

$$\mu_m \leqslant 1 + O\left(\sqrt{\frac{\log\log m}{\log m}}\right).$$

This completes the proof of the theorem. A similar proof works for variants of the Turán-Kubilius inequality.

Finally, Kubilius has informed me privately that he can prove $\lambda < 1{,}764$ [unpublished].

References

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