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A MAPPING THEOREM FOR LOGARITHMIC AND INTEGRATION-BY- PARTS OPERATORS

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ABSTRACT. Suppose U is a set, $\mathbf{F}$ is a field of subsets of U, $\mathfrak{p}_{AB}$ is the set of all bounded real-valued finitely additive functions defined on $\mathbf{F}$, and W is a collection of functions from $\mathbf{F}$ into $\exp(\mathbf{R})$, closed under multiplication, each element of which has range union bounded and bounded away from 0. Let $\mathcal{P}$ denote the set to which T belongs iff T is a function from W into $\mathfrak{p}_{AB}$ such that if each of α and β is in W and V is in $\mathbf{F}$, then the following integrals exist and the following "integration-by-parts" equation holds:

$$\int_V \alpha(I)T(\beta)(I) + \int_V \beta(I)T(\alpha)(I) = T(\alpha\beta)(V).$$

Let $\mathfrak{L}$ denote the set to which S belongs iff S is a function from W into $\mathfrak{p}_{AB}$ such that if each of α and β is in W, then the integral $\int_U \alpha(I)S(\beta)(I)$ exists and the following "logarithmic" equation holds: $S(\alpha\beta) = S(\alpha) + S(\beta)$. It is shown that $\{(T, S)\colon T \text{ in } \mathcal{P},\ S = \{(\alpha, \int (1/\alpha)T(\alpha))\colon \alpha \text{ in } W\}\}$ is a one-one mapping from $\mathcal{P}$ onto $\mathfrak{L}$.

1. **Introduction**. Suppose $[a, b]$ is a number interval and for each function f from $[a, b]$ into $\mathbf{R}$, we let $T(f)([p, q]) = f(q) - f(p)$ for all subintervals $[p, q]$ of $[a, b]$. For suitable functions g and h, then, the "standard" integration-by-parts formula has the following form, in terms of T:

$$\int_a^b gT(h)(I) + \int_a^b hT(g)(I) = T(gh)([a, b]).$$

It is the purpose of this paper to consider the above identity in an abstract setting and establish a one-one-onto mapping theorem that holds with respect to the set of "integration-by-parts" operators (see below) defined on a certain collection of bounded set functions, and the "logarithmic" operators (see below) defined on the same collection. We suppose that $\mathbf{F}$ is a field of subsets of a set U, $\mathfrak{p}_B$ is the set of all functions from $\mathbf{F}$ into $\exp(\mathbf{R})$ with bounded range union, and $\mathfrak{p}_{AB}$ is the set of all bounded finitely additive functions from $\mathbf{F}$ into $\mathbf{R}$.

Now suppose that $W \subseteq \mathfrak{p}_B$ and if each of α and β is in W, then $\alpha\beta$ is in W and $\inf\{|x|\colon x \text{ in range union of } \alpha\} > 0$. We let $\mathcal{P}$ denote the set to which T belongs iff T is a function from W into $\mathfrak{p}_{AB}$ such that if each of α and β is in W and V is in $\mathbf{F}$, then each of the integrals (see §2) written immediately below

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exists and the following equation is satisfied:

$$\int_V \alpha(I)T(\beta)(I) + \int_V \beta(I)T(\alpha)(I) = T(\alpha\beta)(V).$$

We let $\mathfrak{L}$ denote the set to which S belongs iff S is a function from W into $\mathfrak{p}_{AB}$ such that if each of α and β is in W, then $\int_U \alpha(I)S(\beta)(I)$ exists and

$$S(\alpha\beta) = S(\alpha) + S(\beta).$$

We prove the following mapping theorem (§3):

THEOREM 3.1. *Let* X *denote* $\{(T, S)$: T *in* $\mathcal{P}$, $S = \{(\gamma, \int(1/\gamma)T(\gamma))$: γ *in* $W\}\}$. *Then* X *is a one-one mapping from* $\mathcal{P}$ *onto* $\mathfrak{L}$.

2. **Preliminary theorems and definitions.** We refer the reader to [2] for the notion of integral used throughout this paper. We also refer the reader to [2] for a discussion of Kolmogoroff's [3] notion of differential equivalence and its more immediate implications about the existence and equivalence of various integrals. We make the observation that if each of α and β is a function from **F** into exp(**R**) and $\int_U \beta(I)$ exists and $\alpha(I) \subseteq \beta(I)$ for all I in **F**, then for all V in **F**, $\int_V \alpha(I)$ exists and is $\int_V \beta(I)$. We make the further observation that if γ is in $\mathfrak{p}_B$, θ is in $\mathfrak{p}_{AB}$ and $\int_U \gamma(I)\theta(I)$ exists, then $\int \gamma\theta$ is in $\mathfrak{p}_{AB}$ (and, more particularly, in Lip($\int|\theta|$)). We state below extensions of two previous interval function theorems of the author [1]. The proofs carry over with only minor modifications and we therefore omit them.

THEOREM 2.A.1. *If each of* α *and* β *is in* $\mathfrak{p}_B$, θ *is in* $\mathfrak{p}_{AB}$ *and each of* $\int_U \alpha(I)\theta(I)$ *and* $\int_U \beta(I)\theta(I)$ *exists, then* $\int_U \alpha(I)\beta(I)\theta(I)$ *exists.*

THEOREM 2.A.2. *If* α *is in* $\mathfrak{p}_B$, $\inf\{|x|$: x *in range union of* $\alpha\} > 0$, θ *is in* $\mathfrak{p}_{AB}$ *and* $\int_U \alpha(I)\theta(I)$ *exists, then* $\int_U (1/\alpha(I))\theta(I)$ *exists.*

We end this section with the remark that in §3, when the existence of an integral or the equivalence of an integral to an integral is an easy consequence of the material of this section, the integral under discussion need only be written or the equivalence assertion made, and the proof left to the reader.

3. **The mapping theorem.** We prove Theorem 3.1, as stated in the introduction.

PROOF OF THEOREM 3.1. First, suppose T is in $\mathcal{P}$. Let S denote X(T). Clearly, S is a function from W into $\mathfrak{p}_{AB}$. Suppose each of α and β is in W. Then, for each V in **F**, by Theorems 2.A.1 and 2.A.2 we have the following existence and equality:

$$
\begin{aligned}
S(\alpha\beta)(V) &= \int_V \left[\frac{1}{(\alpha(I)\beta(I))} \right] T(\alpha\beta)(I) \\
&= \int_V \left[\frac{1}{(\alpha(I)\beta(I))} \right] \left[\int_I \alpha(J)T(\beta)(J) + \int_I \beta(J)T(\alpha)(J) \right] \\
&= \int_V \left[\frac{1}{(\alpha(I)\beta(I))} \right] \left[\alpha(I)T(\beta)(I) + \beta(I)T(\alpha)(I) \right] \\
&= \int_V \left[\frac{1}{(\alpha(I)\beta(I))} \right] \alpha(I)T(\beta)(I) \\
&\quad + \int_V \left[\frac{1}{(\alpha(I)\beta(I))} \right] \beta(I)T(\alpha)(I) \\
&= \int_V \left(\frac{1}{\beta(I)} \right) T(\beta)(I) + \int_V \left(\frac{1}{\alpha(I)} \right) T(\alpha)(I) \\
&= S(\beta)(V) + S(\alpha)(V).
\end{aligned}
$$

Furthermore, for each V in $\mathbf{F}$, since each of $\int_V \alpha(I)T(\beta)(I)$ and, by Theorem 2.A.2, $\int_V (1/\beta(I))T(\beta)(I)$ exists, it follows from Theorem 2.A.1 that we have the following existence and equality:

$$
\begin{aligned}
\int_V \alpha(I) \left(\frac{1}{\beta(I)} \right) T(\beta)(I) &= \int_V \alpha(I) \left[\int_I \left(\frac{1}{\beta(J)} \right) T(\beta)(J) \right] \\
&= \int_V \alpha(I)S(\beta)(I).
\end{aligned}
$$

Therefore S is in $\mathfrak{L}$.

Now suppose S is in $\mathfrak{L}$. Let T denote $\{(\gamma, \int \gamma S(\gamma)) \colon \gamma \text{ in } W\}$. We show that T is in $\mathcal{P}$. Clearly T is a function from W into $\mathfrak{p}_{AB}$. Furthermore, by Theorem 2.A.1, if each of α and β is in W and V is in $\mathbf{F}$, then we have the following existence and equality:

$$
\int_V \alpha(I)\beta(I)S(\beta)(I) = \int_V \alpha(I) \int_I \beta(J)S(\beta)(J) = \int_V \alpha(I)T(\beta)(I).
$$

Suppose each of α and β is in W and V is in $\mathbf{F}$. Then

$$
\begin{aligned}
T(\alpha\beta)(V) &= \int_V \alpha(I)\beta(I)S(\alpha\beta)(I) = \int_V \alpha(I)\beta(I)\left[S(\alpha)(I) + S(\beta)(I) \right] \\
&= \int_V \beta(I) \int_I \alpha(J)S(\alpha)(J) + \int_V \alpha(I) \int_I \beta(J)S(\beta)(J) \\
&= \int_V \beta(I)T(\alpha)(I) + \int_V \alpha(I)T(\beta)(I).
\end{aligned}
$$

Therefore T is in $\mathcal{P}$. Now, if γ is in W and V is in $\mathbf{F}$, then

$$\mathrm{X}(T)(\gamma)(V) = \int_V \left(\frac{1}{\gamma(I)}\right) T(\gamma)(I) = \int_V \left(\frac{1}{\gamma(I)}\right) \int_I \gamma(J) S(\gamma)(J)$$

$$= \int_V \left(\frac{1}{\gamma(I)}\right) \gamma(I) S(\gamma)(I) = \int_V S(\gamma)(I) = S(\gamma)(V).$$

We have therefore shown that the range of X is $\mathfrak{L}$. Now suppose that each of T_1 and T_2 is in $\mathcal{P}$ and $\mathrm{X}(T_1) = \mathrm{X}(T_2)$. If γ is in W and V is in $\mathbf{F}$, then

$$T_1(\gamma)(V) = \int_V \left(\frac{1}{\gamma(I)}\right) \gamma(I) T_1(\gamma)(I) = \int_V \left(\frac{1}{\gamma(I)}\right) \int_I \gamma(J) T_1(\gamma)(J)$$

$$= \int_V \left(\frac{1}{\gamma(I)}\right) \mathrm{X}(T_1)(\gamma)(I) = \int_V \left(\frac{1}{\gamma(I)}\right) \mathrm{X}(T_2)(\gamma)(I)$$

$$= \int_V \left(\frac{1}{\gamma(I)}\right) \int_I \gamma(J) T_2(\gamma)(J) = \int_V \left(\frac{1}{\gamma(I)}\right) \gamma(I) T_2(\gamma)(I)$$

$$= T_2(\gamma)(V).$$

Therefore $T_1 = T_2$.

Therefore X is a one-one mapping from $\mathcal{P}$ onto $\mathfrak{L}$.

References

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