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ON GENERIC ASYMPTOTIC STABILITY OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

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ABSTRACT. The asymptotic stability of the zero solution of the differential equation ($*$) $x' = Ax + f(x)$ is studied, when the pertubation f is in a given complete metric space $\mathfrak{M}$. It is known that the zero solution of ($*$) is asymptotically stable whenever f is in a certain proper subset $\mathfrak{N} \subset \mathfrak{M}$. It is shown that, while $\mathfrak{N}$ is of Baire first category in $\mathfrak{M}$, on the contrary the set $\mathfrak{M}_0$ of all those f for which the zero solution of ($*$) is asymptotically stable is a proper residual subset of $\mathfrak{M}$.

1. **Introduction.** Denote by E a Banach space with norm $\|\cdot\|$. We shall consider the linear stationary equation

$$x' = Ax \qquad (' = d/dt) \tag{L}$$

where $A: E \to E$ is a linear bounded operator whose spectrum lies in the interior of the left halfplane, so that

$$\|e^{At}\| \leqslant ke^{-\alpha t} \quad \text{for all } t \in \mathbf{R}^+ \ (\alpha > 0,\ 1 \leqslant k < +\infty). \tag{P}$$

We associate with (L) the perturbed equation

$$x' = Ax + f(x)$$

in which the perturbing term f is supposed belonging to some metric space to be specified.

Denote by B_r the open ball in E with center the origin and radius $r > 0$. Define

$$\mathfrak{M} = \{f: B_r \to E \mid f(0) = 0,\ \|f(x) - f(y)\| \leqslant \gamma_f\|x - y\|,\ \gamma_f k/\alpha \leqslant 1\},$$

$$\mathfrak{N} = \{f \in \mathfrak{M} \mid \gamma_f k/\alpha < 1\}.$$

$\mathfrak{M}$, endowed with the distance $\|f - g\| = \sup\{\|f(x) - g(x)\|: x \in B_r\}$, becomes a complete metric space. Observe that $\mathfrak{M}$ is convex.

Under the hypotheses made over A the zero solution of (L) is asymptotically stable. This property remains valid for (P) provided that the perturbing term f is sufficiently small, in particular if $f \in \mathfrak{N}$ [5, p. 504] but it is, in general, lost if it is assumed $f \in \mathfrak{M}$. In this case all one can say is that the origin is merely stable for (P).

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If E is a Hilbert space the set $\mathfrak{N}$ turns out to be small (in the sense of the Baire category) with respect to $\mathfrak{M}$, that is $\mathfrak{N}$ is of first category in $\mathfrak{M}$. Denote by $\mathfrak{M}_0$ the set of all $f \in \mathfrak{M}$ such that the origin is asymptotically stable for (P). Clearly $\mathfrak{M}_0 \supset \mathfrak{N}$. We might then expect that $\mathfrak{M}_0$ is only a bit larger than $\mathfrak{N}$ itself, namely that $\mathfrak{M}_0$ is still of first category in $\mathfrak{M}$. In the present note it will be shown just the opposite. The set $\mathfrak{M}_0$ is in fact residual in $\mathfrak{M}$ (see Theorem 2). This result can be read in another way: with respect to the class $\mathfrak{M}$ of admissible perturbations, the hypothesis $f \in \mathfrak{N}$, though sufficient to guarantee the asymptotic stability of the zero solution of (P), is far away from being a necessary condition as well. To prove the aforementioned theorem we shall use, in the appropriate form, some ideas which go back to Orlicz [4] and which have been further developed in a number of recent papers (see [7] and [1], [2]).

2. **Preliminary lemmas.** In the sequel to emphasize the presence of the perturbation f in the differential equation (P) we shall denote this equation, for a given perturbation f, by $[A, f]$. We agree that saying $[A, f]$ is stable (asymptotically stable) means that the zero solution of $[A, f]$ is stable (asymptotically stable).

REMARK. If $f \in \mathfrak{M}$ and $x_0 \in B_r$, $[A, f]$ has a unique solution $x^f(\cdot\,; x_0)$ which satisfies $x^f(0; x_0) = x_0$ [3]. It follows from the proof of the next lemma that, for every $f \in \mathfrak{M}$,

$$x_0 \in B_{r/k} \Rightarrow \|x^f(t; x_0)\| \leqslant r \quad \text{for all } t \in \mathbf{R}^+,$$

that is each solution of $[A, f]$ which starts in $B_{r/k}$ is defined for all $t \in \mathbf{R}^+$ and does not leave B_r.

LEMMA 1. *For every $f \in \mathfrak{M}$, $[A, f]$ is stable.*

PROOF. Let $f \in \mathfrak{M}$ and $\varepsilon > 0$ ($\varepsilon \leqslant r$). We wish to show that there is $\delta > 0$ ($\delta \leqslant r$) such that

$$\|x_0\| < \delta \Rightarrow \|x^f(t; x_0)\| < \varepsilon \quad \text{for all } t \in \mathbf{R}^+.$$

By Lagrangia formula

$$x^f(t; x_0) = e^{At}x_0 + \int_0^t e^{A(t-s)} f\big(x^f(s; x_0)\big)\, ds$$

and

$$\|x^f(t; x_0)\| \leqslant \|e^{At}\|\, \|x_0\| + \int_0^t \|e^{A(t-s)}\|\, \big\|f\big(x^f(s; x_0)\big)\big\|\, ds$$

$$\leqslant ke^{-\alpha t}\|x_0\| + \gamma k \int_0^t e^{-\alpha(t-s)} \|x^f(s; x_0)\|\, ds.$$

Multiplying both sides of the last inequality by $e^{\alpha t}$, hence using Gronwall's inequality we get

$$\|x^f(t; x_0)\| e^{\alpha t} \leqslant k\|x_0\| e^{\gamma k t}$$

that is

$$\|x^f(t; x_0)\| \leq k\|x_0\|e^{(\gamma k - \alpha)t}.$$

Since $\gamma k - \alpha \leq 0$ the claim follows at once if we let $\delta < \varepsilon / k$.

EXAMPLE. Associate with the stationary equation $x' = -ax$ $(a > 0)$ the perturbed equation given by

$$x' = -ax + bx \tag{2}$$

where b is a real number. Since in this example $k = 1$ and $\alpha = a$, the perturbation term bx is certainly in $\mathfrak{M}$ provided that $|b| \leq a$. Notice that, if $b = -a$, the origin is asymptotically stable for (2) while, if $b = a$, the origin is only stable (but not asymptotically stable).

LEMMA 2. *The set* $\mathfrak{N}$ *is dense in* $\mathfrak{M}$.

PROOF. Let $f \in \mathfrak{M}$ and denote by γ the corresponding Lipschitz constant. Let $\varepsilon > 0$. Set $g_\lambda = \lambda f$, where $0 \leq \lambda < 1$, and observe that g_λ has Lipschitz constant equal to $\lambda\gamma$, thus $g_\lambda \in \mathfrak{N}$. Then

$$\|g_\lambda - f\| = \sup_{x \in B_r} \|\lambda f(x) - f(x)\| \leq (1 - \lambda)\alpha r$$

and $\|g_\lambda - f\| < \varepsilon$ if $\lambda > 1 - \varepsilon/\alpha r$.

Denote by $S(f, \varepsilon)$ the open ball in $\mathfrak{M}$ with center f and radius $\varepsilon > 0$.

LEMMA 3. *Let* $g \in \mathfrak{N}$ *and* $\varepsilon > 0$ $(\varepsilon \leq r)$. *Then there exists* $\delta = \delta_g(\varepsilon) > 0$ *such that, for every* $x_0 \in B_{r/k}$,

$$f \in S(g, \delta) \Rightarrow \|x^f(t; x_0) - x^g(t; x_0)\| < \varepsilon \quad \text{for all } t \in \mathbf{R}^+,$$

where $x^f(\cdot\,; x_0)$ *and* $x^g(\cdot\,; x_0)$ *are solutions of* $[A, f]$ *and* $[A, g]$ *respectively, with initial point* x_0.

PROOF. After the Remark any solution of $[A, f]$ with $f \in \mathfrak{M}$, which starts in $B_{r/k}$ remains in B_r for every $t \in \mathbf{R}^+$. From (1) and the analogous equation for $x^g(\cdot\,; x_0)$ we obtain

$$\begin{aligned}\|x^f(t; x_0) - x^g(t; x_0)\| &\leq \int_0^t \|e^{A(t-s)}\| \, \|f(x^f(s; x_0)) - g(x^f(s; x_0))\| \, ds \\ &\quad + \int_0^t \|e^{A(t-s)}\| \, \|g(x^f(s; x_0)) - g(x^g(s; x_0))\| \, ds \\ &\leq \delta k \int_0^t e^{-\alpha(t-s)} \, ds + \gamma k \int_0^t e^{-\alpha(t-s)} \|x^f(s; x_0) - x^g(s; x_0)\| \, ds,\end{aligned}$$

where γ is the Lipschitz constant of g. Then

$$\|x^f(t; x_0) - x^g(t; x_0)\| e^{\alpha t} \leq \frac{\delta k}{\alpha} e^{\alpha t} + \gamma k \int_0^t \|x^f(s; x_0) - x^g(s; x_0)\| e^{\alpha s} \, ds$$

and, by Gronwall's inequality,

$$\|x^f(t; x_0) - x^g(t; x_0)\| \leq \frac{\delta k}{\alpha} \left[e^{\alpha t} + \gamma k \int_0^t e^{\alpha s} e^{\gamma k(t-s)} \, ds \right]$$

which furnishes

$$\|x^f(t;x_0) - x^g(t;x_0)\| \leqslant \delta \frac{k}{\alpha - \gamma k}, \quad \text{for all } t \in \mathbf{R}^+.$$

Since $\alpha - \gamma k > 0$ to complete the proof it suffices choosing δ such that $0 < \delta < (\alpha - \gamma k)/k$.

3. Main results.

THEOREM 1. *Let E be a Hilbert space. Then the set $\mathfrak{N}$ is of first category in $\mathfrak{M}$.*

PROOF. Let $\{\gamma_i\}$, $0 < \gamma_i < k/\alpha$, be an increasing sequence of reals such that $\gamma_i \to k/\alpha$ as $i \to \infty$. Define

$$\mathfrak{N}_i = \{f \in \mathfrak{M} \mid \|f(x) - f(y)\| \leqslant \gamma_i\|x - y\|\}.$$

It is clear that $\mathfrak{N}_i$ is closed and $\mathfrak{N} = \bigcup_{i=1}^{\infty}\mathfrak{N}_i$. We claim that

$$\operatorname{int} \mathfrak{N}_i = \varnothing, \qquad i = 1, 2, \ldots.$$

Otherwise, for some i, there exist $f \in \mathfrak{N}_i$ and $\varepsilon > 0$ such that $S(f, \varepsilon) \subset \mathfrak{N}_i$. We shall prove that in this sphere there exists at least one function $g \in \mathfrak{M}$ which is not in $\mathfrak{N}_i$.

Choose η such that $0 < \eta < \min\{r, \varepsilon k/2\alpha\}$ and set $\sigma = (\alpha - \gamma_i k)\eta/2\alpha$. Define

$$\tilde{g}(x) = \begin{cases} \alpha x/k, & x \in B_\sigma, \\ f(x), & x \in B_r \setminus B_\eta. \end{cases}$$

We shall show that $\tilde{g}$ satisfies the Lipschitz condition

$$\|\tilde{g}(x) - \tilde{g}(y)\| \leqslant \alpha\|x - y\|/k \quad \text{on } B_\sigma \cup (B_r \setminus B_\eta).$$

To see this it is sufficient to verify the above inequality for $x \in B_\sigma$ and $y \in B_r \setminus B_\eta$. For such choice of x and y we have

$$\begin{aligned} \|\tilde{g}(y) - \tilde{g}(x)\| &= \|f(y) - \alpha x/k\| \leqslant \alpha\|x\|/k + \|f(y) - f(0)\| \\ &\leqslant \alpha\|x\|/k + \gamma_i\|y\| \leqslant \alpha\sigma/k + \gamma_i\|y\| \\ &\leqslant \alpha(\|y\| - \sigma)/k \leqslant \alpha\|y - x\|/k. \end{aligned}$$

By virtue of a theorem of Valentine [6] there exists an extension g of $\tilde{g}$ which is defined all over B_r and is there Lipschitzian with the same constant α/k. Obviously $g \in S(f, \varepsilon)$ and $g \notin \mathfrak{N}_i$, a contradiction. This completes the proof.

Define

$$\mathfrak{M}_0 = \{f \in \mathfrak{M} \mid [A, f] \text{ is asymptotically stable}\}.$$

THEOREM 2. *The set $\mathfrak{M}_0$ is residual in $\mathfrak{M}$.*

PROOF. Define $V\colon \mathfrak{M} \to \mathbf{R}^+$ by

$$V(f) = \sup_{x_0 \in B_{r/k}} \varlimsup_{t \to \infty} \|x^f(t; x_0)\|,$$

where $x^f(\cdot;\, x_0)$ is the solution of $[A, f]$ through x_0. The functional V has the properties: (a) $V(f) = 0$ for each $f \in \mathfrak{N}$; (b) If $f \in \mathfrak{M}$ and $V(f) = 0$, then $[A, f]$ is asymptotically stable; (c) Let $g \in \mathfrak{N}$. Then for every $f \in S(g, \delta_g(1/n))$ we have $V(f) \leqslant 1/n$.

Indeed, (a) and (b) are obvious while (c) follows immediately from the inequality

$$\|x^f(t;\, x_0)\| \leqslant \|x^f(t;\, x_0) - x^g(t;\, x_0)\| + \|x^g(t;\, x_0)\|$$

by virtue of Lemma 3 and the fact that $[A, g]$ is asymptotically stable.

Now define

$$\mathfrak{M}_1 = \bigcap_{n=1}^{\infty} \bigcup_{g \in \mathfrak{N}} S\left(g, \delta_g\left(\frac{1}{n}\right)\right).$$

Observe that, for every $f \in \mathfrak{M}_1$, $V(f) = 0$ thus $[A, f]$ is asymptotically stable. Clearly $\mathfrak{M}_1$ is dense in $\mathfrak{M}$ since $\mathfrak{M}_1 \supset \mathfrak{N}$. On the other hand $\mathfrak{M}_1$ is a G_δ subset of $\mathfrak{M}$ and, $\mathfrak{M}$ being a complete metric space, it follows that $\mathfrak{M}_1$ is a residual set in $\mathfrak{M}$. Since $\mathfrak{M}_0 \supset \mathfrak{M}_1$, the proof is complete.

4. **Concluding remarks.** The results of the preceding section suggest the following definition of stability.

Consider the perturbed equation $[A, f]$ and suppose that the perturbing term f is in some complete metric space $\mathfrak{F}$.

DEFINITION. *The zero solution of* $[A, f]$, $f \in \mathfrak{F}$, *is said to be generically stable (generically asymptotically stable) on* $\mathfrak{F}$ *if the set of all* $f \in \mathfrak{F}$ *for which* $[A, f]$ *admits the zero solution and this solution is stable (asymptotically stable) is residual in* $\mathfrak{F}$.

Then Theorem 2 can be rephrased as follows. *The zero solution of* $[A, f]$, $f \in \mathfrak{M}$ *is generically asymptotically stable on* $\mathfrak{M}$.

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