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$H^\infty(R) + AP$ IS CLOSED

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ABSTRACT. Let $H^\infty(R)$ be the space of functions on the real line R which are boundary functions of functions bounded and analytic in the upper half-plane and let AP denote the space of uniformly almost periodic functions on R. We show that $H^\infty(R) + AP$ is closed and is not an algebra.

Introduction. Let H^∞ denote the space of essentially bounded measurable functions on the unit circle T whose negative Fourier coefficients vanish and let C denote the continuous functions on T. Sarason [4] has shown that $H^\infty + C$ is closed. More recently Rudin [3] has shown how this result and natural generalisations of it follow from a simple abstract Banach space closure theorem. We shall use Rudin's theorem to prove a similar result for functions on the real line where C is replaced by AP, the space of uniformly almost periodic functions. The key ingredient of Rudin's theorem is a set of maps which form a bounded approximate identity for the C component. Here we shall use the existence of an invariant mean on the bounded functions on R to define a set of maps on $L^\infty(R)$ which is a bounded approximate identity for AP corresponding to the classical convergence of Bochner-Fejér polynomials to uniformly almost periodic functions.

We ask whether certain abstract spaces of type $H^\infty + C$ are always closed. Such a result would include our result, Sarason's result and several of Rudin's generalisations.

THEOREM 1 (RUDIN [3]). *Let Y and Z be closed subspaces of a Banach space X and let Φ be a family of bounded linear maps Λ on X such that*

(a) Λ *maps* X *into* Y, $\Lambda \in \Phi$,

(b) Λ *maps* Z *into* Z, $\Lambda \in \Phi$,

(c) $\sup\{\|\Lambda\|; \Lambda \in \Phi\} < \infty$,

(d) *for each* $y \in Y$ *and* $\varepsilon > 0$ *there exists* $\Lambda \in \Phi$ *such that* $\|\Lambda y - y\| < \varepsilon$.

Then $Y + Z$ *is closed.*

NOTATION. The usual Fejér kernel on R is given by

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$$K_n(t) = \frac{1}{n}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2 = \sum_{|\nu|<n}\left(1 - \frac{|\nu|}{n}\right)e^{-i\nu t} \qquad (n \in N).$$

Let $K_{\mathbf{n},\boldsymbol{\beta}}(t) = \prod_{1\leqslant i\leqslant p} K_{n_i}(\beta_i t)$, where $\beta_1, \beta_2, \ldots, \beta_p$ are real numbers and $n_i \in N$ $(1 \leqslant i \leqslant p)$. These are the Bochner-Fejér kernels [1, p. 46]. For $f \in AP$ the mean

$$M(f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(s)\,ds$$

always exists [1, p. 12]. Thus for $f \in AP$ we can consider

$$\sigma_{\mathbf{n},\boldsymbol{\beta}}(x) = M(f_{-x}K_{\mathbf{n},\boldsymbol{\beta}})$$

where $f_{-x}(t) = f(t + x)$, $x \in R$. Since M is translation invariant a simple computation shows that

$$\sigma_{\mathbf{n},\boldsymbol{\beta}}(x) = \sum_{\substack{|\nu_1|<n_1\\ \vdots \\ |\nu_p|<n_p}} \left(1 - \frac{|\nu_1|}{n_1}\right)\cdots\left(1 - \frac{|\nu_p|}{n_p}\right) M\{f(t)e^{-i\langle\boldsymbol{\nu},\boldsymbol{\beta}\rangle t}\}e^{i\langle\boldsymbol{\nu},\boldsymbol{\beta}\rangle x},$$

where $\langle \boldsymbol{\nu}, \boldsymbol{\beta}\rangle = \nu_1\beta_1 + \nu_2\beta_2 + \cdots + \nu_p\beta_p$. We call $\sigma_{\mathbf{n},\boldsymbol{\beta}}$ the Bochner-Fejér polynomial for f corresponding to $\mathbf{n}$, $\boldsymbol{\beta}$.

The following approximation theorem can be found in [1, p. 50].

THEOREM 2. *For each f in AP there exist Bochner-Fejér polynomials for f converging uniformly to f.*

Extending invariant means. Let $B(R)$ be the space of bounded complex functions on the real line. For f in $B(R)$ and x in R let f_x be the translate of f defined by $f_x(y) = f(y - x)$ $(y \in R)$.

DEFINITION. If W is a subspace of $B(R)$ (or $L^\infty(R)$) which is closed under translation and contains the constant functions, then an invariant mean on W is a linear functional m such that $\|m\| = m(1) = 1$ and $m(f_x) = m(f)$ $(f \in W, x \in R)$.

The following theorem, whose proof we omit, is a simple corollary of Theorem 17.5 in [2].

THEOREM 3. *There exists an invariant mean on $B(R)$.*

The next theorem allows us to extend the mean M to an invariant mean on $L^\infty(R)$. With this state we construct the maps Λ necessary for the application of Rudin's theorem.

THEOREM 4. *Let W be a closed translation invariant subspace of $L^\infty(R)$ containing the constant functions and let M_0 be an invariant mean on W. Then there exists an invariant mean $\tilde{M}$ on $L^\infty(R)$ extending M_0.*

PROOF. By Theorem 3 there exists an invariant mean, m say, on $B(R)$. Let

M_1 be a state on $L^{\infty}(R)$ extending M_0. For $g \in L^{\infty}(R)$ define $\phi_g \in B(R)$ by $\phi_g(y) = M_1(g_y)$ for y in R. Now define $\tilde{M}$ on $L^{\infty}(R)$ by $\tilde{M}(g) = m(\phi_g)$. It is straightforward to verify that $\tilde{M}$ is an invariant mean on $L^{\infty}(R)$ extending M_0.

The main result. Let $H^{\infty}(R)$ be the space of boundary value functions of bounded analytic functions in the open upper half-plane.

THEOREM 5. *$H^{\infty}(R) + AP$ is closed.*

PROOF. Let W be the space of functions f in $L^{\infty}(R)$ such that the mean $M(f)$ exists. Then M is an invariant mean on the translation invariant subspace W. By Theorem 4 there exists an invariant mean $\tilde{M}$ on $L^{\infty}(R)$ extending M.

Define $\Lambda_{\mathbf{n},\boldsymbol{\beta}}$ on $L^{\infty}(R)$ by

$$(\Lambda_{\mathbf{n},\boldsymbol{\beta}} f)(x) = \tilde{M}(f_{-x} K_{\mathbf{n},\boldsymbol{\beta}}) \qquad (f \in L^{\infty}(R)).$$

Since $\tilde{M}$ is a state and $K_{\mathbf{n},\boldsymbol{\beta}}(t) \geqslant 0$ $(t \in R)$,

$$|(\Lambda_{\mathbf{n},\boldsymbol{\beta}} f)(x)| \leqslant \|f\| \tilde{M}\{K_{\mathbf{n},\boldsymbol{\beta}}\} = \|f\|. \tag{1}$$

Since $\tilde{M}$ is translation invariant it follows that

$$(\Lambda_{\mathbf{n},\boldsymbol{\beta}} f)(x) = \sum_{\substack{|\nu_1| < n_1 \\ \vdots \\ |\nu_p| < n_p}} \left(1 - \frac{|\nu_1|}{n_1}\right) \cdots \left(1 - \frac{|\nu_p|}{n_p}\right) \tilde{M}\left\{f(t) e^{-i\langle \boldsymbol{\nu}, \boldsymbol{\beta}\rangle t}\right\} e^{i\langle \boldsymbol{\nu}, \boldsymbol{\beta}\rangle x}. \tag{2}$$

Thus

$$\Lambda_{\mathbf{n},\boldsymbol{\beta}} \colon L^{\infty}(R) \to AP. \tag{3}$$

For f in AP, by Theorem 2, there exist $\Lambda^{(k)}$ in $\{\Lambda_{\mathbf{n},\boldsymbol{\beta}}\}$ for $k = 1, 2, \ldots,$ such that $\Lambda^{(k)} f \to f$ in $L^{\infty}(R)$ as $k \to \infty$. In view of this and (1) and (3) above, to show that $H^{\infty}(R) + AP$ is closed it suffices to show that $\Lambda_{\mathbf{n},\boldsymbol{\beta}} \colon H^{\infty}(R) \to H^{\infty}(R)$. For then we can apply Theorem 1 with $L^{\infty}(R)$ in place of X, $H^{\infty}(R)$ in place of Z, AP in place of Y and $\{\Lambda_{\mathbf{n},\boldsymbol{\beta}}\}$ in place of Φ. In fact by (2) it will be sufficient to show that $\tilde{M}\{f(t)e^{i\lambda t}\} = 0$ for $\lambda > 0$ and f in $H^{\infty}(R)$.

For f in $H^{\infty}(R)$ and $\lambda > 0$ we have

$$\int_{-T}^{T} f(t) e^{i\lambda t}\, dt = \int_C f(z) e^{i\lambda z}\, dz,$$

where C is the semicircular contour from $-T$ to T in the upper half-plane. Thus

$$\begin{aligned} \left| \int_{-T}^{T} f(t) e^{i\lambda t}\, dt \right| &\leqslant \|f\|_{\infty} \int_C |e^{i\lambda z}|\, |dz| \\ &\leqslant \|f\|_{\infty} \pi/\lambda. \end{aligned}$$

It follows that $M\{f(t)e^{i\lambda t}\}$ exists and is zero, completing the proof.

The following similar result has been proved by Sarason in [5]. If BUC denotes the space of bounded uniformly continuous functions on R then $H^\infty(R) + BUC$ is closed. In fact he shows that $H^\infty(R) + BUC$ is the closed linear span of functions of the form $e^{i\lambda x}f(x)$ where λ is real and f belongs to $H^\infty(R)$. In particular $H^\infty(R) + BUC$ is an algebra (as is $H^\infty + C$). It is now easy to see that $H^\infty(R) + AP$ cannot be an algebra. For if it were then since $e^{-i\lambda x}$ is in AP, by Theorem 5 and the above we see that $H^\infty(R) + AP = H^\infty(R) + BUC$. This implies that BUC is contained in $H^\infty(R) + AP$ which is false since $M\{f\}$ need not exist for f in BUC.

Problem. Let F be a commutative family of normal operators on a Hilbert space. Let H^∞ denote the weakly closed algebra generated by F and let C be the C^*-algebra generated by F. Is $H^\infty + C$ closed?

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