ON VECTOR STATES AND SEPARABLE C*-ALGEBRAS

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ABSTRACT. It is proved that the set of states on a separable C*-subalgebra of the Calkin algebra may be simultaneously extended to a set of equivalent, orthogonal, pure states on the Calkin algebra.

Let $\mathfrak{A}$ denote a separable C*-algebra of operators acting on a separable Hilbert space $\mathcal{H}$ and suppose that $\mathfrak{A}$ contains the identity. In [2] Glimm proved that the weak*-closure of the set of vector states on $\mathfrak{A}$ (i.e., states on $\mathfrak{A}$ of the form $\omega_x(A) = (Ax, x)$, where $A \in \mathfrak{A}$ and $x$ is a unit vector in $\mathcal{H}$) contains the set $\mathcal{S}(\mathfrak{A})$ of all states on $\mathfrak{A}$ which annihilate $\mathfrak{A} \cap \mathcal{K}(\mathcal{H})$. ($\mathcal{K}(\mathcal{H})$ denotes the compact operators acting on $\mathcal{H}$.) Voiculescu used this result in [3] in the proof of his noncommutative Weyl-von Neumann theorem. In this note Voiculescu's theorem shall be used to obtain a stronger version of Glimm's result: There is a sequence $\{\omega_n\}$ of vector states, induced by an orthonormal set of vectors in $\mathcal{H}$, such that $\mathcal{S}(\mathfrak{A})$ is contained in the weak*-closure of $\{\omega_n\}$. (It should be noted that Glimm's theorem holds without any separability assumptions so that the theorem to be proved here is stronger only in the separable case.)

This theorem, together with a theorem from [1], yields a somewhat surprising corollary: There is a set $S$ consisting of equivalent, orthogonal, pure states on $\mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$, such that every state in $S(\mathfrak{A})$ is a restriction of a state in $S$. In particular, if $f$ is any state on a separable C*-subalgebra of the Calkin algebra, $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, then there is a pure state $g$ on the Calkin algebra which extends $f$.

To prove the theorem, note that since $\mathfrak{A}$ is separable, $S(\mathfrak{A})$ is weak*-metrizable and compact and so contains a countable dense set, say $\{f_n\}$.

Let $p$ denote the canonical homomorphism of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra. Then each state $f_n$ determines a state $g_n$ on $p(\mathfrak{A})$ such that $f_n = g_n \circ p$. Let $\{\pi_n, \mathcal{H}_n, x_n\}$ denote the G.N.S. representation of $p(\mathfrak{A})$ constructed from $g_n$. Then $\pi_n$ is a *-homomorphism of $p(\mathfrak{A})$ into $\mathcal{B}(\mathcal{H}_n)$ and $f_n(A) = g_n \circ p(A) = (\pi_n \circ p(A)x_n, x_n)$ for each $A$ in $\mathfrak{A}$. Let $\pi$ denote the representation of $p(\mathfrak{A})$ obtained by taking the direct sum of the $\pi_n$'s, so that $\pi$ maps $p(\mathfrak{A})$ into $\mathcal{B}(\Sigma \oplus \mathcal{H}_n)$. By Voiculescu's theorem, there is a unitary transformation $U$ of

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\[ K \rightarrow K \oplus \sum \oplus K_n \] such that \( A = U^*(A \oplus \pi \circ p(A))U \in K(K) \) for all \( A \) in \( \mathcal{A} \). Write \( e_n = U^*x_n \) for \( n = 1, 2, \ldots \). Then \( \{e_n\} \) is an orthonormal sequence in \( K \) and the vector states \( \omega_n = \omega_{e_n}, n = 1, 2, \ldots \), have the desired property. Indeed, if \( f \in \mathcal{S}(\mathcal{A}) \), choose an infinite subsequence \( \{f_n\} \) of \( \{f_n\} \) which converges to \( f \) in the weak*-topology. Fix \( A \in \mathcal{A} \). Then

\[
f(A) = \lim_j f_n(A) = \lim_j g_{\pi_n} \circ p(A) = \lim_j \left( \pi_n \circ p(A)x_n, x_n \right)
\]

\[
= \lim_j \left( U^*(A \oplus \pi \circ p(A))Ue_n, e_n \right) = \lim_j \omega_n(A) + \lim\left( Ke_n, e_n\right),
\]

where \( K \) is the compact operator \( U^*(A \oplus \pi \circ p(A))U - A \). Since \( \{e_n\} \) converges weakly to zero and \( K \) is compact, \( \|K e_n\| \to 0 \) as \( j \to \infty \). Hence, \( f(A) = \lim_j \omega_n(A) \) for all \( A \) in \( \mathcal{A} \), as desired.

To prove the corollary, choose a sequence \( \{\omega_n\} \) of vector states induced by an orthonormal sequence \( \{e_n\} \) such that each \( f \) in \( \mathcal{S}(\mathcal{A}) \) is the weak*-limit of a subsequence of the \( \omega_n \)'s. Fix a free ultrafilter \( \mathcal{U} \) on the natural numbers \( \mathbb{N} \) and define a state \( g \) on \( \mathcal{B}(\mathcal{K}) \) by \( g(T) = \lim_{\mathcal{U}} \omega_n(T) \). For each permutation \( \alpha \) of \( \mathcal{N} \) define a unitary operator \( U_\alpha \) on \( K \) by \( U_\alpha e_n = e_{\alpha(n)}, n = 1, 2, \ldots \), and define the state \( g_\alpha \) on \( \mathcal{B}(\mathcal{K}) \) by \( g_\alpha(T) = g(U_\alpha^*TU_\alpha) = \lim_{\mathcal{U}} \omega_{\alpha(n)}(T) \). (Adding vectors if necessary, we may assume that \( \{e_n\} \) is a basis for \( K \).) Then the set \( \mathcal{S} = \{ g_\alpha : \alpha \text{ is a permutation of } \mathcal{N} \} \) has the desired properties. Indeed, by [1, Corollary 3] \( g \), and hence each \( g_\alpha \), is a pure state on \( \mathcal{B}(\mathcal{K}) \) (because \( \{e_n\} \) is an orthonormal sequence). Thus, \( \mathcal{S} \) consists of equivalent pure states. Further, if \( \alpha \) and \( \beta \) are permutations of \( \mathcal{N} \) such that \( g_\alpha \) and \( g_\beta \) are distinct elements of \( \mathcal{S} \), then there are disjoint subsets \( \sigma \) and \( \tau \) of \( \mathcal{N} \) such that \( \alpha^{-1}(\sigma) \subseteq \mathcal{U} \) and \( \beta^{-1}(\tau) \subseteq \mathcal{U} \). If \( D \) is defined by \( D e_n = e_n \) for \( n \in \sigma \), \( De_n = -e_n \) for \( n \in \tau \) and \( De_n = 0 \) otherwise, then \( D \in \mathcal{B}(\mathcal{K}) \), \( \|D\| = 1 \) and \( g_\alpha(D) - g_\beta(D) = 2 \). Hence, \( \|g_\alpha - g_\beta\| = 2 \) and the elements of \( \mathcal{S} \) are orthogonal. Finally, if \( f = \lim_{\mathcal{U}} \omega_n \) is a state in \( \mathcal{S}(\mathcal{A}) \), then for some permutation \( \alpha \) of \( \mathcal{N} \), \( \alpha^{-1}\{n_1, n_2, \ldots \} \subseteq \mathcal{U} \) and

\[
g_\alpha(A) = \lim_{\mathcal{U}} \omega_{\alpha(n)}(A) = \lim_j \omega_n(A) = f(A)
\]

for \( A \in \mathcal{A} \). The proof is complete.

Note that the choice of \( \mathcal{S} \) in the proof above is far from unique. In fact, there are \( 2^c \) disjoint sets of states on \( \mathcal{B}(\mathcal{K}) \) which have the desired properties. (As usual, \( c \) denotes the cardinality of the continuum.) Furthermore, by altering the proof somewhat, it is possible to choose a set \( \mathcal{S}' \) of disjoint (i.e., inequivalent) pure states on \( \mathcal{B}(\mathcal{K}) \) such that \( \mathcal{S}' \mid_\mathcal{A} = \mathcal{S}(\mathcal{A}) \).

As an example, take \( \mathcal{A} \) to be an isometric isomorphic image of \( C(0, 1) \), the continuous functions on the unit interval, in \( \mathcal{B}(\mathcal{K}) \). Then \( \mathcal{A} \cap \mathcal{K}(\mathcal{K}) = \{0\} \) and \( \mathcal{S}(\mathcal{A}) \) is the entire set of states on \( \mathcal{A} \). Hence, every state on \( \mathcal{A} \) (including integration) extends to a pure state on \( \mathcal{B}(\mathcal{K}) \).

In conclusion, it seems worth noting that the fact that states in \( \mathcal{S}(\mathcal{A}) \) extend to pure states on \( \mathcal{B}(\mathcal{K}) \) may be proved without recourse to Voiculescu's theorem. Indeed, by a theorem of Wils [4], if \( f \in \mathcal{S}(\mathcal{A}) \), then \( f = \lim_{\mathcal{U}} \omega_{x_n} \).
where $\mathcal{U}$ is a free ultrafilter on the natural numbers and $(x_n)$ is a sequence of unit vectors in $\mathcal{K}$ such that $\lim_{n}(x_n, y) = 0$ for all $y$ in $\mathcal{K}$. Straightforward arguments using the separability of $\mathcal{K}$ can then be used to show that $f = \lim_{n} \omega_{x_n}$, where $(e_n)$ is an orthonormal sequence in $\mathcal{K}$. The proof is completed, as before, by invoking Corollary 3 of [1].

REFERENCES


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