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ON VECTOR STATES AND SEPARABLE C^*-ALGEBRAS

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ABSTRACT. It is proved that the set of states on a separable C^*-subalgebra of the Calkin algebra may be simultaneously extended to a set of equivalent, orthogonal, pure states on the Calkin algebra.

Let $\mathfrak{A}$ denote a separable C^*-algebra of operators acting on a separable Hilbert space $\mathcal{H}$ and suppose that $\mathfrak{A}$ contains the identity. In [2] Glimm proved that the weak*-closure of the set of vector states on $\mathfrak{A}$ (i.e., states on $\mathfrak{A}$ of the form $\omega_x(A) = (Ax, x)$, where $A \in \mathfrak{A}$ and x is a unit vector in $\mathcal{H}$) contains the set $\mathcal{S}(\mathfrak{A})$ of all states on $\mathfrak{A}$ which annihilate $\mathfrak{A} \cap \mathcal{K}(\mathcal{H})$. ($\mathcal{K}(\mathcal{H})$ denotes the compact operators acting on $\mathcal{H}$.) Voiculescu used this result in [3] in the proof of his noncommutative Weyl-von Neumann theorem. In this note Voiculescu's theorem shall be used to obtain a stronger version of Glimm's result: *There is a sequence* $\{\omega_n\}$ *of vector states, induced by an orthonormal set of vectors in* $\mathcal{H}$, *such that* $\mathcal{S}(\mathfrak{A})$ *is contained in the weak*-closure of* $\{\omega_n\}$. (It should be noted that Glimm's theorem holds without any separability assumptions so that the theorem to be proved here is stronger only in the separable case.)

This theorem, together with a theorem from [1], yields a somewhat surprising corollary: *There is a set* $\mathcal{E}$ *consisting of equivalent, orthogonal, pure states on* $\mathfrak{B}(\mathcal{H})$, *the bounded linear operators on* $\mathcal{H}$, *such that every state in* $\mathcal{S}(\mathfrak{A})$ *is a restriction of a state in* $\mathcal{E}$. *In particular, if f is any state on a separable* C^*-*subalgebra of the Calkin algebra*, $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, *then there is a pure state g on the Calkin algebra which extends f.*

To prove the theorem, note that since $\mathfrak{A}$ is separable, $s(\mathfrak{A})$ is weak*-metrizable and compact and so contains a countable dense set, say $\{f_n\}$.

Let p denote the canonical homomorphism of $\mathfrak{B}(\mathcal{H})$ onto the Calkin algebra. Then each state f_n determines a state g_n on $p(\mathfrak{A})$ such that $f_n = g_n \circ p$. Let $\{\pi_n, \mathcal{H}_n, x_n\}$ denote the G.N.S. representation of $p(\mathfrak{A})$ constructed from g_n. Then π_n is a *-homomorphism of $p(\mathfrak{A})$ into $\mathfrak{B}(\mathcal{H}_n)$ and $f_n(A) = g_n \circ p(A) = (\pi_n \circ p(A)x_n, x_n)$ for each A in $\mathfrak{A}$. Let π denote the representation of $p(\mathfrak{A})$ obtained by taking the direct sum of the π_n's, so that π maps $p(\mathfrak{A})$ into $\mathfrak{B}(\Sigma \oplus \mathcal{H}_n)$. By Voiculescu's theorem, there is a unitary transformation U of

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$\mathcal{H}$ onto $\mathcal{H} \oplus \Sigma \oplus \mathcal{H}_n$ such that $A - U^*(A \oplus \pi \circ p(A))U \in \mathcal{K}(\mathcal{H})$ for all A in $\mathfrak{A}$. Write $e_n = U^* x_n$ for $n = 1, 2, \ldots$. Then $\{e_n\}$ is an orthonormal sequence in $\mathcal{H}$ and the vector states $\omega_n = \omega_{e_n}$, $n = 1, 2, \ldots$, have the desired property. Indeed, if $f \in \mathcal{S}(\mathfrak{A})$, choose an infinite subsequence $\{f_{n_j}\}$ of $\{f_n\}$ which converges to f in the weak*-topology. Fix A in $\mathfrak{A}$. Then

$$f(A) = \lim_j f_{n_j}(A) = \lim_j g_{n_j} \circ p(A) = \lim_j \left(\pi_{n_j} \circ p(A) x_{n_j}, x_{n_j}\right)$$
$$= \lim_j \left(U^*(A \oplus \pi \circ p(A)) U e_{n_j}, e_{n_j}\right) = \lim_j \omega_{n_j}(A) + \lim_j \left(K e_{n_j}, e_{n_j}\right),$$

where K is the compact operator $U^*(A \oplus \pi \circ p(A))U - A$. Since $\{e_{n_j}\}$ converges weakly to zero and K is compact, $\|K e_{n_j}\| \to 0$ as $j \to \infty$. Hence, $f(A) = \lim_j \omega_{n_j}(A)$ for all A in $\mathfrak{A}$, as desired.

To prove the corollary, choose a sequence $\{\omega_n\}$ of vector states induced by an orthonormal sequence $\{e_n\}$ such that each f in $\mathcal{S}(\mathfrak{A})$ is the weak*-limit of a subsequence of the ω_n's. Fix a free ultrafilter $\mathcal{U}$ on the natural members $\mathcal{N}$ and define a state g on $\mathcal{B}(\mathcal{H})$ by $g(T) = \lim_{\mathcal{U}} \omega_n(T)$. For each permutation α of N define a unitary operator U_α on $\mathcal{H}$ by $U_\alpha e_n = e_{\alpha(n)}$, $n = 1, 2, \ldots,$ and define the state g_α on $\mathcal{B}(\mathcal{H})$ by $g_\alpha(T) = g(U_\alpha^* T U_\alpha) = \lim_{\mathcal{U}} \omega_{\alpha(n)}(T)$. (Adding vectors if necessary, we may assume that $\{e_n\}$ is a basis for $\mathcal{H}$.) Then the set $\mathcal{G} = \{g_\alpha : \alpha \text{ is a permutation of } \mathcal{N}\}$ has the desired properties. Indeed, by [1, Corollary 3] g, and hence each g_α, is a pure state on $\mathcal{B}(\mathcal{H})$ (because $\{e_n\}$ is an orthonormal sequence). Thus, $\mathcal{G}$ consists of equivalent pure states. Further, if α and β are permutations of $\mathcal{N}$ such that g_α and g_β are distinct elements of $\mathcal{G}$, then there are disjoint subsets σ and τ of $\mathcal{N}$ such that $\alpha^{-1}(\sigma) \in \mathcal{U}$ and $\beta^{-1}(\tau) \in \mathcal{U}$. If D is defined by $De_n = e_n$ for $n \in \sigma$, $De_n = -e_n$ for $n \in \tau$ and $De_n = 0$ otherwise, then $D \in \mathcal{B}(\mathcal{H})$, $\|D\| = 1$ and $g_\alpha(D) - g_\beta(D) = 2$. Hence, $\|g_\alpha - g_\beta\| = 2$ and the elements of $\mathcal{G}$ are orthogonal. Finally, if $f = \lim_j \omega_{n_j}$ is a state in $\mathcal{S}(\mathfrak{A})$, then for some permutation α of $\mathcal{N}$, $\alpha^{-1}(\{n_1, n_2, \ldots\}) \in \mathcal{U}$ and

$$g_\alpha(A) = \lim_{\mathcal{U}} \omega_{\alpha(n)}(A) = \lim_j \omega_{n_j}(A) = f(A)$$

for $A \in \mathfrak{A}$. The proof is complete.

Note that the choice of $\mathcal{G}$ in the proof above is far from unique. In fact, there are 2^c disjoint sets of states on $\mathcal{B}(\mathcal{H})$ which have the desired properties. (As usual, c denotes the cardinality of the continuum.) Furthermore, by altering the proof somewhat, it is possible to choose a set $\mathcal{G}'$ of disjoint (i.e., inequivalent) pure states on $\mathcal{B}(\mathcal{H})$ such that $\mathcal{G}'|_{\mathfrak{A}} = \mathcal{S}(\mathfrak{A})$.

As an example, take $\mathfrak{A}$ to be an isometric isomorphic image of $C(0, 1)$, the continuous functions on the unit interval, in $\mathcal{B}(\mathcal{H})$. Then $\mathfrak{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$ and $\mathcal{S}(\mathfrak{A})$ is the entire set of states on $\mathfrak{A}$. Hence, *every* state on $\mathfrak{A}$ (including integration) extends to a pure state on $\mathcal{B}(\mathcal{H})$.

In conclusion, it seems worth noting that the fact that states in $\mathcal{S}(\mathfrak{A})$ extend to pure states on $\mathcal{B}(\mathcal{H})$ may be proved without recourse to Voiculescu's theorem. Indeed, by a theorem of Wils [4], if $f \in \mathcal{S}(\mathfrak{A})$, then $f = \lim_{\mathcal{U}} \omega_{x_n}$

where $\mathfrak{U}$ is a free ultrafilter on the natural numbers and $\{x_n\}$ is a sequence of unit vectors in $\mathcal{H}$ such that $\lim_{\mathfrak{U}}(x_n, y) = 0$ for all y in $\mathcal{H}$. Straightforward arguments using the separabilility of $\mathfrak{A}$ can then be used to show that $f = \lim_n \omega_{e_n}$, where $\{e_n\}$ is an orthonormal sequence in $\mathcal{Q}$. The proof is completed, as before, by invoking Corollary 3 of [1].

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