GENERIC MORSE-SMALE DIFFEOMORPHISMS
HAVE ONLY TRIVIAL SYMMETRIES

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ABSTRACT. The purpose of this paper is to prove that for a $C^1$-generic
Morse-Smale diffeomorphism $f$, the set of symmetries of $f$, $Z(f)$, is equal to
$\{j^k | k \in \mathbb{Z}\}$.

1. Introduction. Let $M$ be a compact connected $C^\infty$-manifold without
boundary. Let $\text{Diff}(M)$ be the set of $C^1$-diffeomorphisms of $M$ with
$C^1$-topology. Let $\text{MS}$ denote the open set of all Morse-Smale diffeomor-
phisms of $M$ in $\text{Diff}(M)$ [5]. For $f \in \text{Diff}(M)$ we say $g \in \text{Diff}(M)$ is a
symmetry of $f$ iff $f \circ g = g \circ f$. Then the centralizer $Z(f)$ of $f$ is the set of all
symmetries of $f$. Clearly, $f^k$ is a symmetry of $f$ for any $k \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of
integers). We call such symmetries trivial symmetries. A proper symmetry is a
symmetry which is not trivial. The following question is posed by N. Kopell
[4] and J. Palis [6].

Is the set of diffeomorphisms without proper symmetry generic in $\text{Diff}(M)$?

In this paper, we shall prove the following theorem which gives an affirmati-
ve solution of the conjecture in MS.

THEOREM. It is $C^1$-generic in $\text{MS}$ that $f$ has no proper symmetry.

The referee pointed out that the work of Boyd Anderson in [1] is closely
related to ours. See also [2].

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error in the proof of Lemma 5.1.

2. Proof of the Theorem. Choose a riemannian metric on $M$. For a tangent
vector $v$ we let $\|v\|$ denote the length of $v$ by this riemannian metric. Let
$J^1(M)$ denote the 1-jet space on $M$ ($= \bigcup_{x,y \in M} L(T_xM, T_yM)$), and $\pi_i$
$J^1(M) \rightarrow M$ ($i = 1, 2$) denote the projections, i.e., $\pi_1(\alpha) = x$, $\pi_2(\alpha) = y$
for $\alpha \in L(T_xM, T_yM)$. For $m \in \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers) we define
$J^1(M : m)$ as the set of all $\alpha$'s in $J^1(M)$ such that $1/m < \|\alpha\| < m$, for any
tangent vector $v$ of norm 1. We fix a countable basis $\Theta$ of the topology of $M$
and a countable dense subset $M^*$ of $M$. Let

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proper symmetry.
Definition 2.1. Let \( \{ B(U, x_0, m) \} \) be the family defined as follows; for \((U, x_0, m) \in \Gamma \), \( B(U, x_0, m) \) is the set of all Morse-Smale diffeomorphism \( f \)'s which have a mapping \( S: O_f(x_0) \to J^1(M: m) \) satisfying the following conditions;

(i) \( \pi_1 \circ S = \text{identity} \),
(ii) \( \pi_2 \circ S(x_0) \not\in O_f(U) \) and
(iii) \( f \circ S(x) = S(f(x)) \circ T_{f} \), for any \( x \in O_f(x_0) \), where \( O_f(x_0) \) denotes the orbit of \( x_0 \).

Remark. Condition (iii) implies that 
\[
f(\pi_2 \circ S(x)) = \pi_2 \circ S(f(x))
\]
for any \( x \in O_f(x_0) \). Therefore (ii) implies that 
(ii)' \( \pi_2 \circ S(x) \not\in O_f(U) \) for any \( x \in O_f(x_0) \).

In order to prove the theorem it is sufficient to verify Propositions 2.1, 2.2 and 2.3.

Proposition 2.1. If \( f \in MS \) has a proper symmetry, then \( f \) is contained in one of \( B(U, x_0, m) \)'s.

Proposition 2.2. Each \( B(U, x_0, m) \) is closed.

Proposition 2.3. Each \( B(U, x_0, m) \) has no interior point.

In the following sections, we shall prove these propositions.

3. Proof of Proposition 2.1. We need the following lemma:

Lemma 3.1. Let \( f \in MS \) and \( g \in \text{Diff}(M) \). Suppose that \( g(x) \in O_f(x) \) for any \( x \in M^* - \text{per} f \). Then \( g \) is a trivial symmetry of \( f \).

Proof. Let \( A_k = \{ x \in M - \text{per} f | g(x) = f^k(x) \} \). Then the family \( \{ A_k \}_{k \in \mathbb{Z}} \) is disjoint, each \( A_k \) is closed in \( M - \text{per} f \), and \( M^* - \text{per} f \) is contained in \( \bigcup A_k \). Let \( V \) be a connected open set such that \( \text{Cl}(V) \subset M - \text{per} f \). Since \( V \) is open, \( V \cap M^* \) is dense in \( \text{Cl}(V) \). We claim that \( \text{Cl}(V) \subset A_k \) for some \( k \in \mathbb{Z} \). Let \( Q_\epsilon(x) = \{ y \in M | d(x, y) < \epsilon \} \). Since \( \text{Cl}(V) \) consists only of wandering points, there exists \( \epsilon > 0 \) such that \( f^n(Q_\epsilon(x)) \cap f^m(Q_\epsilon(x)) = \emptyset \) for any distinct integers \( n \) and \( m \), and any \( x \in \text{Cl}(V) \) because of the compactness of \( \text{Cl}(V) \). Then there exists \( \epsilon' > 0 \) such that \( g(Q_\epsilon(x)) \subset f^k(Q_\epsilon(x)) \) for any \( x \in A_k \cap V \). Notice that \( \epsilon' \) depends on \( k \) but this presents no problem. Let \( x^* \in A_k \cap V \cap M^* \). Since \( g(Q_{\epsilon'}(x^*)) \) is contained in \( f^k(Q_{\epsilon'}(x^*)) \) and \( M^* \cap V \subset \bigcup A_k \), \( Q_{\epsilon'}(x^*) \cap M^* \) is contained in \( A_k \). This implies that \( Q_{\epsilon'}(x^*) \subset A_k \), and since \( V \) is connected, \( V \) is contained in \( A_k \).

Let \( B_1, \ldots, B_t \) denote the connected components of \( M - \text{per} f \). We can choose a connected open set \( V \) for any points \( x \) and \( y \) in \( B_i \) such that \( x, y \in \text{Cl}(V) \subset B_i \), so \( B_i \) is contained in \( A_k \) for some \( k \in \mathbb{Z} \).

If \( \dim(M) > 2 \), then \( M - \text{per} f \) is connected, and equivalently, \( M - \text{per} f = B_1 \); hence \( g \) is trivial.
If \( \dim(M) = 1 \), equivalently, \( M = S^1 \), \( g \) can be nontrivial only if there is a periodic point \( p \) such that \( g = f^k \) in the right neighbourhood of \( p \) and \( g = f^{k'} \) in the left neighbourhood of \( p \). But this contradicts with the assumption that \( p \) is hyperbolic. Hence \( g \) is trivial.

**Proof of Proposition 2.1.** Let \( f \) be a Morse-Smale diffeomorphism with a proper symmetry \( g \). By Lemma 3.1, we can choose a point \( x_0 \in M^* - \text{per}f \) such that \( g(x_0) \notin O_f(x_0) \). We show that we can choose \( U \in \Theta \) such that \( x_0 \in U \) and \( O_f(g(x_0)) \cap O_f(U) = \emptyset \). Since \( x_0 \notin \text{nonwandering set} \), it is not in either \( \alpha- \) or \( \omega- \) limit set of \( g(x_0) \); since it is by hypothesis not in \( O_f(g(x_0)) \), we can conclude that the point \( x_0 \) does not belong to the closed set \( \text{Cl}(O_f(g(x_0))) \), so there exists a neighbourhood \( U \in \Theta \) of \( x_0 \) disjoint from \( O_f(g(x_0)) \); but then \( O_f(U) \cap O_f(g(x_0)) = \emptyset \). Let us choose \( m \in \mathbb{N} \) such that \( m > \max(||Tg_v||, ||Tg^{-1}v||) \) for any \( v \in TM \) of norm 1. Define \( S(x) \) by \( S(x) = T_{x^g}g \) for \( x \in O_f(x_0) \). It is clear that \( f \in B(U, x_0, m) \).

4. **Proof of Proposition 2.2.** Suppose that a sequence \( \{f_n\} \) of diffeomorphisms of \( B(U, x_0, m) \) converges to \( f \in \text{Diff}(M) \). For each \( f_n \) we choose a map \( S_n \), which satisfies the conditions of Definition 2.1. Let us define a map \( S \) for \( f \) as follows. Since \( J^1(M; m) \) is compact, the sequence \( S_n(x_0) \) has cluster points. Define \( S(x_0) \) to be one of the cluster points. Then \( S(x_0) \in J^1(M; m) \) and \( \pi_1 \circ S(x_0) = x_0 \). We define

\[
S: O_f(x_0) \rightarrow J^1(M; m)
\]

by

\[
S(x_k) = Tf^k \circ S(x_0) \circ T_f^{-k}\big|T_{x_k}M
\]

for \( x_k = f^k(x_0) \), where \( T_{x_k}M \) denotes the tangent plane on \( x_k \). Clearly \( S \) satisfies conditions (i) and (iii) of Definition 2.1. We check that \( S \) satisfies condition (ii). First notice that

\[
\pi_2 \circ S(x_0) \subset \text{Cl}\left( \{ f_n^k \circ \pi_2 \circ S_n(x_0) \}_{n \in \mathbb{N}} \right)
\]

since \( \pi_2 \circ S(x_0) \) is a cluster point of \( \{ \pi_2 \circ S_n(x_0) \}_{n \in \mathbb{N}} \) and \( \{f_n^k\}_{n \in \mathbb{N}} \) converges to \( f^k \) for any fixed \( k \). But since any \( f_n^k(\pi_2 \circ S_n(x_0)) \) is not in \( U \), neither is \( f^k(\pi_2 \circ S(x_0)) \) in \( U \); this implies condition (ii) of Definition 2.1.

5. **Proof of Proposition 2.3.** Let \( \rho: \mathbb{R}^q \rightarrow \mathbb{R}^q \) be a \( C^\infty \)-function with the following properties:

(i) \( \max(||\rho||, ||D\rho||) < 1 \),
(ii) \( \rho(0) = 0 \) and \( D\rho(0) = \text{identity} \), and
(iii) \( \rho(x) = 0 \) for any \( ||x|| > 1 \).

**Definition 5.1.** A number sequence \( \{a_n\} \to 0 \) is called of exponential type iff for some \( a > 0 \) and \( K > 0 \), \( a^n/K < a_n < Ka^n \) for any \( n \in \mathbb{N} \).

**Lemma 5.1.** Let \( L: \mathbb{R}^q \rightarrow \mathbb{R}^q \) be a semisimple linear contraction, i.e., \( L \) has a matrix \( (a_{ij}) \) such that
\[a_{2i-1, 2i-1} = |\lambda_i| \cos \theta_i, \quad a_{2i-1, 2i} = -|\lambda_i| \sin \theta_i,\]
\[a_{2i, 2i-1} = |\lambda_i| \sin \theta_i, \quad a_{2i, 2i} = |\lambda_i| \cos \theta_i,\]
for \(1 < i < q',\) and \(a_{ij} = \lambda_j\) for \(2q' < i,\) and the others = 0 for some \(0 < q' < q/2\) and \(0 < |\lambda_j| < 1.\)

Let \(B = \{x \in \mathbb{R}^q : ||x|| < 1\}\) and \(e_1 = (1, 0, \ldots, 0).\) Suppose that \(0 \neq x_0 \in B\) and let \(U \subset B\) be an open neighbourhood of \(x_0\) such that \(L^n(U) \cap L^m(U) = \emptyset\) for any distinct integers \(n\) and \(m.\) Then for any \(\varepsilon > 0\) there is a \(C^1\)-local diffeomorphism \(f : B \rightarrow \mathbb{R}^q\) such that

(i) \(\max(\|f - L\|, \|Df - DL\|) < \varepsilon,\)
(ii) \(f|B - O_L(U) = L|B - O_L(U),\)
(iii) the sequence \(\{\|D^n(f(x_0)e_i\|\}\) is not of exponential type.

**Proof.** Let \(x_n = L^n x_0.\) We choose \(\delta(n)\) such that \(0 < \delta(n) < \min(||x_n||/2, d(x_n, B - L^n(U)))\) for \(n \in \mathbb{N}.\) Define \(\rho_n\) by
\[\rho_n(x) = \varepsilon \delta(n)\rho(x/\delta(n)).\]

We define \(f\) by
\[f|B - O_L(U) = L|B - O_L(U)\]
and
\[f(x) = Lx + \rho_{n+1}(L(x - x_n))/(n + 1)\]
for \(x \in L^n(U).\) Then \(f\) is well defined as a continuous mapping and of class \(C^1\) on \(B - \{0\}.\) Since
\[\|Df(x) - L\| < \|D\rho_{n+1}\| \cdot \|L\|/(n + 1) < \varepsilon/(n + 1)\]
for \(x \in L^n(U),\) then \(Df(x) \rightarrow L\) as \(x \rightarrow 0,\) and hence \(f\) is continuously differentiable at \(0.\) Clearly \(C^1\)-distance between \(f\) and \(L\) is less than \(\varepsilon,\) so \(f\) is a diffeomorphism provided that \(\varepsilon\) is sufficiently small.

Now we check that the convergence of the sequence \(\|D^n(f(x_0)e_i)\|\) is not of exponential type. Let \(e^n_1 = D^n(f(x_0)e_1).\) Then
\[\|e^n_{i+1}\| = \|Df(x_n)e^n_i\| = (|\lambda_1| + |\lambda_i|\varepsilon/(n + 1))\|e^n_i\|\]
and
\[\|D^n(x_0)e_i\| = \|e^n_i\| = \prod_{i=1}^{n} |\lambda_i|(1 + \varepsilon/i).\]
Notice that the convergence of the sequence \(\{\|e^n_i\|\}\) is of exponential type iff the sequence \(\sum_{i=1}^{n} (1 + \varepsilon/i)\) converges. But since a sequence \(\sum_{i=1}^{n} (1 + a_n)\) converges iff \(\sum a_n\) does [4, Theorem 3, p. 94], the sequence \(\{\|e^n_i\|\}\) is not of exponential type.

**Proof of Proposition 2.3.** Let \(f \in B(U, x_0, m).\) We approximate \(f\) by a Morse-Smale diffeomorphism which is not in \(B(U, x_0, m).\) We first approximate \(f\) by \(f_1\) which has the following properties:
(i) there exists a sink \(p\) of \(f_1\) such that \(x_0 \in W^s(p),\)
(ii) for any periodic point \(q\) of \(f_1,\) setting \(l(q)\) equal to the period of \(q\) under
$f_1, f_1^{(u)}$ is semisimple linear in some chart.

Since $B(U, x_0, m) \subset B(U', x_0, m)$ for any neighbourhood $U'$ of $x_0$ such that $U' \in \Theta$ and $U' \subset U$, we can suppose, without loss of generality, $U \subset W^s(p)$ and $f^n(U) \cap f^{n'}(U) = \emptyset$ for any distinct integers $n$ and $n'$. By (ii), if \{$||Tf_1^nu||\$} converges to 0, then the convergence is of exponential type for any nonzero tangent vector $v$. Let us perturb $f_1$ near $p$, as in Lemma 5.1, by taking $f_1$ as the linear map $L$. Then we get an approximation $f_2$ of $f_1$ with a tangent vector $v$ on $x_0$ such that the sequence \{$||Tf_2^nv||\$} is not of exponential type. Then we show that $f_2 \notin B(U, x_0, m)$. Suppose the contrary, and consider the sequence
\[
\left\{ ||S(x_n)Tf_2^nv|| \right\} = \left\{ ||Tf_2^nS(x_0)v|| \right\}
\]
where $S$ is a map for $f_2$ in Definition 2.1. Since $S(x_0)v$ is a tangent vector on $M - O_2(U)$ and $f_2$ coincides with $f_1$ on $O_2(U) = O_2'(U)$, the sequence \{$||S(x_n)Tf_2^nv||\$} = \{$||Tf_2^nS(x_0)v||\$} is of exponential type. Since
\[
||Tf_2^nv|| / m < ||S(x_n)Tf_2^nv|| < m ||Tf_2^nv||
\]
by Definition 2.1, so
\[
||S(x_n)Tf_2^nv|| / m < ||Tf_2^nv|| < m ||S(x_n)Tf_2^nv||.
\]
But \{$||S(x_n)Tf_2^nv||\$} is of exponential type; then \{$||Tf_2^nv||\$} is also of exponential type, a contradiction.

References


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