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GENERIC MORSE-SMALE DIFFEOMORPHISMS HAVE ONLY TRIVIAL SYMMETRIES

YOSHIO TOGAWA

ABSTRACT. The purpose of this paper is to prove that for a C^1-generic Morse-Smale diffeomorphism f, the set of symmetries of f, $Z(f)$, is equal to $\{f^k | k \in \mathbf{Z}\}$.

1. **Introduction**. Let M be a compact connected C^∞-manifold without boundary. Let Diff(M) be the set of C^1-diffeomomrphisms of M with C^1-topology. Let MS denote the open set of all Morse-Smale diffeomorphisms of M in Diff(M) [5]. For $f \in \text{Diff}(M)$ we say $g \in \text{Diff}(M)$ is *a symmetry of f* iff $f \circ g = g \circ f$. Then the centralizer $Z(f)$ of f is the set of all symmetries of f. Clearly, f^k is a symmetry of f for any $k \in \mathbf{Z}$ ($\mathbf{Z}$ is the set of integers). We call such symmetries *trivial symmetries*. *A proper symmetry* is a symmetry which is not trivial. The following question is posed by N. Kopell [4] and J. Palis [6].

Is the set of diffeomorphisms without proper symmetry generic in Diff(M)?

In this paper, we shall prove the following theorem which gives an affirmative solution of the conjecture in MS.

THEOREM. *It is C^1-generic in* MS *that f has no proper symmetry.*

The referee pointed out that the work of Boyd Anderson in [1] is closely related to ours. See also [2].

I would like to thank Professor Hiroshi Noguchi for his kind advice. I would also like to thank the referee for pointing out how to overcome an error in the proof of Lemma 5.1.

2. **Proof of the Theorem**. Choose a riemannian metric on M. For a tangent vector v we let $\|v\|$ denote the length of v by this riemannian metric. Let $J^1(M)$ denote the 1-jet space on M $(= \bigcup_{x,y \in M} L(T_xM, T_yM))$, and π_i: $J^1(M) \to M$ $(i = 1, 2)$ denote the projections, i.e., $\pi_1(\alpha) = x$, $\pi_2(\alpha) = y$ for $\alpha \in L(T_xM, T_yM)$. For $m \in \mathbf{N}$ ($\mathbf{N}$ is the set of natural numbers) we define $J^1(M : m)$ as the set of all α's in $J^1(M)$ such that $1/m \leqslant \|\alpha v\| \leqslant m$, for any tangent vector v of norm 1. We fix a countable basis Θ of the topology of M and a countable dense subset M^* of M. Let

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$$\Gamma = \{(U, x_0, m) | U \in \Theta, x_0 \in M^*, x_0 \in U, m \in \mathbf{N}\}.$$

DEFINITION 2.1. Let $\{B(U, x_0, m)\}_\Gamma$ be the family defined as follows; for $(U, x_0, m) \in \Gamma$, $B(U, x_0, m)$ is the set of all Morse-Smale diffeomorphism f's which have a mapping S: $O_f(x_0) \to J^1(M: m)$ satisfying the following conditions;

(i) $\pi_1 \circ S = $ identity,

(ii) $\pi_2 \circ S(x_0) \not\subset O_f(U)$ and

(iii) $Tf \circ S(x) = S(f(x)) \circ T_x f$, for any x in $O_f(x_0)$, where $O_f(x_0)$ denotes the orbit of x_0.

REMARK. Condition (iii) implies that

$$f(\pi_2 \circ S(x)) = \pi_2 \circ S(f(x))$$

for any $x \in O_f(x_0)$. Therefore (ii) implies that

(ii)′ $\pi_2 \circ S(x) \not\subset O_f(U)$ for any $x \in O_f(x_0)$.

In order to prove the Theorem it is sufficient to verify Propositions 2.1, 2.2 and 2.3.

PROPOSITION 2.1. *If $f \in$ MS has a proper symmetry, then f is contained in one of $B(U, x_0, m)$'s.*

PROPOSITION 2.2. *Each $B(U, x_0, m)$ is closed.*

PROPOSITION 2.3. *Each $B(U, x_0, m)$ has no interior point.*

In the following sections, we shall prove these propositions.

3. **Proof of Proposition 2.1.** We need the following lemma:

LEMMA 3.1. *Let $f \in$ MS and $g \in$ Diff(M). Suppose that $g(x) \in O_f(x)$ for any $x \in M^* -$ per f. Then g is a trivial symmetry of f.*

PROOF. Let $A_k = \{x \in M -$ per $f | g(x) = f^k(x)\}$. Then the family $\{A_k\}_{k \in \mathbf{Z}}$ is disjoint, each A_k is closed in $M -$ per f, and $M^* -$ per f is contained in $\bigcup A_k$. Let V be a connected open set such that $\mathrm{Cl}(V) \subset M -$ per f. Since V is open, $V \cap M^*$ is dense in $\mathrm{Cl}(V)$. We claim that $\mathrm{Cl}(V) \subset A_k$ for some $k \in \mathbf{Z}$. Let $Q_\varepsilon(x) = \{y \in M | d(x, y) < \varepsilon\}$. Since $\mathrm{Cl}(V)$ consists only of wandering points, there exists $\varepsilon > 0$ such that $f^n(Q_\varepsilon(x)) \cap f^{n'}(Q_\varepsilon(x)) = \varnothing$ for any distinct integers n and n', and any $x \in \mathrm{Cl}(V)$ because of the compactness of $\mathrm{Cl}(V)$. Then there exists $\varepsilon' > 0$ such that $g(Q_{\varepsilon'}(x)) \subset f^k(Q_\varepsilon(x))$ for any $x \in A_k \cap V$. Notice that ε' depends on k but this presents no problem. Let $x^* \in A_k \cap V \cap M^*$. Since $g(Q_{\varepsilon'}(x^*))$ is contained in $f^k(Q_\varepsilon(x^*))$ and $M^* \cap V \subset \bigcup A_k$, $Q_{\varepsilon'}(x^*) \cap M^*$ is contained in A_k. This implies that $Q_{\varepsilon'}(x^*) \subset A_k$, and since V is connected, V is contained in A_k.

Let $B_1, \ldots, B_s$ denote the connected components of $M -$ per f. We can choose a connected open set V for any points x and y in B_i such that $x, y \in \mathrm{Cl}(V) \subset B_i$, so B_i is contained in A_k for some $k \in \mathbf{Z}$.

If $\dim(M) \geqslant 2$, then $M -$ per f is connected, and equivalently, $M -$ per $f = B_1$; hence g is trivial.

If $\dim(M) = 1$, equivalently, $M = S^1$, g can be nontrivial only if there is a periodic point p such that $g = f^k$ in the right neighbourhood of p and $g = f^{k'}$ in the left neighbourhood of p. But this contradicts with the assumption that p is hyperbolic. Hence g is trivial.

PROOF OF PROPOSITION 2.1. Let f be a Morse-Smale diffeomorphism with a proper symmetry g. By Lemma 3.1, we can choose a point $x_0 \in M^* - \operatorname{per} f$ such that $g(x_0) \notin O_f(x_0)$. We show that we can choose $U \in \Theta$ such that $x_0 \in U$ and $O_f(g(x_0)) \cap O_f(U) = \varnothing$. Since $x_0 \notin$ nonwandering set, it is not in either α- or ω-limit set of $g(x_0)$; since it is by hypothesis not in $O_f(g(x_0))$, we can conclude that the point x_0 does not belong to the closed set $\mathrm{Cl}(O_f(g(x_0)))$, so there exists a neighbourhood $U \in \Theta$ of x_0 disjoint from $O_f(g(x_0))$; but then $O_f(U) \cap O_f(g(x_0)) = \varnothing$. Let us choose $m \in \mathbf{N}$ such that $m > \max(\|Tgv\|, \|Tg^{-1}v\|)$ for any $v \in TM$ of norm 1. Define $S(x)$ by $S(x) = T_x g$ for $x \in O_f(x_0)$. It is clear that $f \in B(U, x_0, m)$.

4. **Proof of Proposition 2.2.** Suppose that a sequence $\{f_n\}$ of diffeomorphisms of $B(U, x_0, m)$ converges to $f \in \mathrm{Diff}(M)$. For each f_n we choose a map S_n which satisfies the conditions of Definition 2.1. Let us define a map S for f as follows. Since $J^1(M\colon m)$ is compact, the sequence $S_n(x_0)$ has cluster points. Define $S(x_0)$ to be one of the cluster points. Then $S(x_0) \in J^1(M\colon m)$ and $\pi_1 \circ S(x_0) = x_0$. We define

$$S\colon O_f(x_0) \to J^1(M\colon m)$$

by

$$S(x_k) = Tf^k \circ S(x_0) \circ Tf^{-k}|T_{x_k}M$$

for $x_k = f^k(x_0)$, where $T_{x_k}M$ denotes the tangent plane on x_k. Clearly S satisfies conditons (i) and (iii) of Definition 2.1. We check that S satisfies condition (ii). First notice that

$$f^k(\pi_2 \circ S(x_0)) \subset \mathrm{Cl}\Big(\big\{f_n^k(\pi_2 \circ S_n(x_0))\big\}_{n \in \mathbf{N}}\Big)$$

since $\pi_2 \circ S(x_0)$ is a cluster point of $\{\pi_2 \circ S_n(x_0)\}_{n \in \mathbf{N}}$ and $\{f_n^k\}_{n \in \mathbf{N}}$ converges to f^k for any fixed k. But since any $f_n^k(\pi_2 \circ S_n(x_0))$ is not in U, neither is $f^k(\pi_2 \circ S(x_0))$ in U; this implies condition (ii) of Definition 2.1.

5. **Proof of Proposition 2.3.** Let $\rho\colon \mathbf{R}^q \to \mathbf{R}^q$ be a C^∞-function with the following properties:

(i) $\max(\|\rho\|, \|D\rho\|) \leqslant 1$,

(ii) $\rho(0) = 0$ and $D\rho(0) =$ identity, and

(iii) $\rho(x) = 0$ for any $\|x\| \geqslant 1$.

DEFINITION 5.1. A number sequence $\{a_n\} \to 0$ is called *of exponential type* iff for some $a > 0$ and $K > 0$, $a^n/K \leqslant a_n \leqslant Ka^n$ for any $n \in \mathbf{N}$.

LEMMA 5.1. *Let L: $\mathbf{R}^q \to \mathbf{R}^q$ be a semisimple linear contraction, i.e., L has a matrix (a_{ij}) such that*

$$a_{2i-1,2i-1} = |\lambda_i|\cos\theta_i, \qquad a_{2i-1,2i} = -|\lambda_i|\sin\theta_i,$$
$$a_{2i,2i-1} = |\lambda_i|\sin\theta_i, \qquad a_{2i,2i} = |\lambda_i|\cos\theta_i,$$

for $1 \leq i \leq q'$, *and* $a_{ii} = \lambda_i$ *for* $2q' \leq i$, *and the others* $= 0$ *for some* $0 \leq q' \leq q/2$ *and* $0 < |\lambda_i| < 1$.

Let $B = \{x \in \mathbf{R}^q | \, \|x\| < 1\}$ *and* $e_1 = (1, 0, \dots, 0)$. *Suppose that* $0 \neq x_0 \in B$ *and let* $U \in B$ *be an open neighbourhood of* x_0 *such that* $L^n(U) \cap L^{n'}(U) = \emptyset$ *for any distinct integers* n *and* n'. *Then for any* $\varepsilon > 0$ *there is a* C^1*-local diffeomorphism* $f\colon B \to \mathbf{R}^q$ *such that*

(i) $\operatorname{Max}(\|f - L\|, \|Df - DL\|) < \varepsilon$,

(ii) $f|B - O_f(U) = L|B - O_L(U)$,

(iii) *the sequence* $\{\|Df^n(x_0)e_1\|\}$ *is not of exponential type.*

Proof. Let $x_n = L^n X_0$. We choose $\delta(n)$ such that $0 < \delta(n) < \min(\|x_n\|/2, d(x_n, B - L^n(U)))$ for $n \in \mathbf{N}$. Define ρ_n by

$$\rho_n(x) = \varepsilon\delta(n)\rho(x/\delta(n)).$$

We define f by

$$f|B - O_L(U) = L|B - O_L(U)$$

and

$$f(x) = Lx + \rho_{n+1}(L(x - x_n))/(n + 1)$$

for $x \in L^n(U)$. Then f is well defined as a continuous mapping and of class C^1 on $B - \{0\}$. Since

$$\|Df(x) - L\| \leq \|D\rho_{n+1}\| \cdot \|L\|/(n + 1) < \varepsilon/(n + 1)$$

for $x \in L^n(U)$, then $Df(x) \to L$ as $x \to 0$, and hence f is continuously differentiable at 0. Clearly C^1-distance between f and L is less than ε, so f is a diffeomorphism provided that ε is sufficiently small.

Now we check that the convergence of the sequence $\|Df^n(x_0)e_1\|$ is not of exponential type. Let $e_1^n = Df^n(x_0)e_1$. Then

$$\|e_1^{n+1}\| = \|Df(x_n)e_1^n\| = (|\lambda_1| + |\lambda_1|\varepsilon/(n + 1))\|e_1^n\|$$

and

$$\|Df^n(x_0)e_1\| = \|e_1^n\| = \prod_{i=1}^{i=n} |\lambda_1|(1 + \varepsilon/i).$$

Notice that the convergence of the sequence $\{\|e_1^n\|\}$ is of exponential type iff the sequence $\prod_{i=1}^{i=n}(1 + \varepsilon/i)$ converges. But since a sequence $\prod(1 + a_n)$ converges iff Σa_n does [4, Theorem 3, p. 94], the sequence $\{\|e_1^n\|\}$ is not of exponential type.

Proof of Proposition 2.3. Let $f \in B(U, x_0, m)$. We approximate f by a Morse-Smale diffeomorphism which is not in $B(U, x_0, m)$. We first approximate f by f_1 which has the following properties:

(i) there exists a sink p of f_1 such that $x_0 \in W^s(p)$,

(ii) for any periodic point q of f_1, setting $l(q)$ equal to the period of q under

f_1, $f_1^{l(q)}$ is semisimple linear in some chart.

Since $B(U, x_0, m) \subset B(U', x_0, m)$ for any neighbourhood U' of x_0 such that $U' \in \Theta$ and $U' \subset U$, we can suppose, without loss of generality, $U \subset W^s(p)$ and $f_1^n(U) \cap f_1^{n'}(U) = \emptyset$ for any distinct integers n and n'. By (ii), if $\{\|Tf_1^n v\|\}$ converges to 0, then the convergence is of exponential type for any nonzero tangent vector v. Let us perturb f_1 near p, as in Lemma 5.1, by taking f_1 as the linear map L. Then we get an approximation f_2 of f_1 with a tangent vector v on x_0 such that the sequence $\{\|Tf_2^n v\|\}$ is not of exponential type. Then we show that $f_2 \notin B(U, x_0, m)$. Suppose the contrary, and consider the sequence

$$\{\|S(x_n)Tf_2^n v\|\} = \{\|Tf_2^n S(x_0)v\|\}$$

where S is a map for f_2 in Definition 2.1. Since $S(x_0)v$ is a tangent vector on $M - O_{f_2}(U)$ and f_2 coincides with f_1 on $O_{f_1}(U) = O_{f_2}(U)$, the sequence $\{\|S(x_n)Tf_2^n v\|\} = \{\|Tf_2^n S(x_0)v\|\}$ is of exponential type. Since

$$\|Tf_2^n v\|/m \leqslant \|S(x_n)Tf_2^n v\| \leqslant m\|Tf_2^n v\|$$

by Definition 2.1, so

$$\|S(x_n)Tf_2^n v\|/m \leqslant \|Tf_2^n v\| \leqslant m\|S(x_n)Tf_2^n v\|.$$

But $\{\|S(x_n)Tf_2^n v\|\}$ is of exponential type; then $\{\|Tf_2^n v\|\}$ is also of exponential type, a contradiction.

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Department of Mathematics, School of Science and Engineering, Waseda University, Shinjuku, Tokyo, Japan

Current address: Science University of Tokyo, Department of Information Science, Faculty of Science and Technology, Noda City (278), Japan