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THE FIVE LEMMA FOR BANACH SPACES

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ABSTRACT. We prove a version of the five lemma which is useful for the study of boundary value problems for partial differential equations. The results are given in the category $\mathfrak{B}$ of Banach spaces and bounded linear operators, and all conditions are stated modulo an arbitrary ideal of $\mathfrak{B}$. We also show that the results are valid in more general categories.

1. **Introduction.** The five lemma for homomorphisms of abelian groups (Spanier [4, Lemma 4.5.11]) can be divided into two parts, one concluding that a map is a monomorphism and the other that it is an epimorphism. If the groups are taken to be Banach spaces and the homomorphisms are bounded linear operators, then, by the closed graph theorem, the isomorphisms of abelian groups are isomorphisms of Banach spaces. Similarly, an epimorphism whose kernel has a closed complement (as in the case, for example, when the domain space is a Hilbert space) is an operator with a bounded right inverse. However, apart from those with closed complemented range, monomorphisms are not operators with bounded left inverses. This fact restricts the possible applications of the standard five lemma.

In this paper we prove a version of the five lemma for bounded linear operators between Banach spaces; with epimorphisms and monomorphisms replaced, respectively, by operators with bounded right and left inverses modulo any given ideal of bounded linear operators. Further, the conditions of exactness and commutativity are only required modulo the ideal.

We make considerable use of this five lemma in a paper, yet to be published, on elliptic partial differential equations with mixed boundary conditions. Some of these applications are discussed in §4.

In §5, we show that our results are valid in any additive subcategory of the category of abelian groups and homomorphisms.

This work formed part of a Ph.D. thesis prepared at Macquarie university in Australia. I am indebted to Alan McIntosh, my supervisor for that thesis, and wish to express my gratitude for his valuable suggestions and encouragement.

2. **Notation.** Let $\mathfrak{B}$ denote the class of all Banach spaces and $\mathfrak{L}$ the class of all bounded linear operators between members of $\mathfrak{B}$. Let $\mathfrak{M}$ denote any *ideal*

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of $\mathfrak{L}$ (or $\mathfrak{B}$) in the sense that $\mathfrak{M}$ is a nonempty proper subclass of $\mathfrak{L}$ which is closed under composition with members of $\mathfrak{L}$ and under addition.

For example, $\mathfrak{K}$, the class of all compact operators between Banach spaces, is an ideal. So too are $\mathfrak{FR}$, the class of all operators in $\mathfrak{L}$ with finite rank, and $\mathfrak{S}$, the class of strictly singular operators in $\mathfrak{L}$. The trivial ideal of zero operators between members of $\mathfrak{B}$ is denoted 0. Other examples of ideals are found, for example, in Retherford [1].

A subspace of a Banach space will be called *complemented* if it is closed and has a closed complement. Hence, a subspace is complemented if and only if it is the image of a *projection* or idempotent in $\mathfrak{L}$.

An operator $S: X \to Y$ in $\mathfrak{L}$ will be said to have a *left* (*right*) *inverse modulo* $\mathfrak{M}$, say S': $Y \to X$ with $S' \in \mathfrak{L}$, if $I_X - S'S \in \mathfrak{M}$ ($I_Y - SS' \in \mathfrak{M}$). The operator S' is a *pseudo inverse modulo* $\mathfrak{M}$ of S if $S - SS'S \in \mathfrak{M}$.

Note that an operator in $\mathfrak{L}$ with a left inverse modulo $\mathfrak{FR}$ is just a *left semi-Fredholm* (*lsF*) operator in the sense that it is a bounded linear operator with finite dimensional kernel and closed complemented range. Similarly an operator in $\mathfrak{L}$ with a right inverse modulo $\mathfrak{FR}$ is a *right semi-Fredholm* (*rsF*) operator, or a bounded linear operator which has a complemented kernel and a closed range with finite codimension. Results similar to these can be found, for example, in Schechter [2], [3]. An operator in $\mathfrak{L}$ is *Fredholm* if and only if it is lsF and rsF.

If S: $X \to Y$ belongs to $\mathfrak{L}$ and has a pseudo inverse modulo 0, say S', then $I_X - S'S$ is a projection onto the kernel of S, and SS' is a projection onto the range of S. Hence the kernel and range of S are each complemented. It follows that a member of $\mathfrak{L}$ has a pseudo inverse modulo 0 if and only if its kernel and range are complemented.

Suppose now that $S_1: X_0 \to X_1$ and $S_2: X_1 \to S_2$ belong to $\mathfrak{L}$ and have pseudo inverses, S_1' and S_2', respectively, modulo an ideal $\mathfrak{M}$. We define

$$S_1 \triangle S_2 = (I_{X_1} - S_1 S_1')(I_{X_1} - S_2' S_2).$$

This definition depends on the choice of S_1' and S_2'. However, if $S_1 \triangle'' S_2 = (I_{X_1} - S_1 S_1'')(I_{X_1} - S_2'' S_2)$, where S_1'' and S_2'' are also pseudo inverses modulo $\mathfrak{M}$ of S_1 and S_2, respectively, then we have the following result.

LEMMA. $S_1 \triangle S_2 \in \mathfrak{M}$ *if and only if* $S_1 \triangle'' S_2 \in \mathfrak{M}$.

PROOF. With all equalities being understood modulo $\mathfrak{M}$, it follows from the definition of a pseudo inverse modulo $\mathfrak{M}$ that

$$(I_{X_1} - S_1 S_1'') = (I_{X_1} - S_1 S_1'')(I_{X_1} - S_1 S_1'),$$

$$(I_{X_1} - S_2'' S_2) = (I_{X_1} - S_2' S_2)(I_{X_1} - S_2'' S_2),$$

and hence

$$\begin{aligned} S_1 \triangle'' S_2 &= (I_{X_1} - S_1 S_1'')(I_{X_1} - S_2'' S_2) \\ &= (I_{X_1} - S_1 S_1'')(S_1 \triangle S_2)(I_{X_1} - S_2'' S_2). \end{aligned}$$

By symmetry, and the definition of an ideal, the result follows.

With this lemma in mind, we define a sequence

$$\cdots \to X_i \xrightarrow{S_i} X_{i+1} \xrightarrow{S_{i+1}} X_{i+2} \to \cdots$$

of mappings in $\mathfrak{L}$ to be *exact modulo* $\mathfrak{M}$ if for each i, S_i and S_{i+1} have pseudo inverses modulo $\mathfrak{M}$ and $S_{i+1} S_i$ and $S_i \triangle S_{i+1}$ belong to $\mathfrak{M}$. The sequence is called *exact* if the range of S_i is the kernel of S_{i+1} for each i.

We note that if S_i and S_{i+1} have pseudo inverses modulo 0, S_i' and S_{i+1}', respectively, then $I_{X_{i+1}} - S_{i+1}' S_{i+1}$ is a projection onto the kernel of S_{i+1} and $I_{X_{i+1}} - S_i S_i'$ has the range of S_i as its kernel. Hence, using the lemma, $S_i \triangle S_{i+1} = 0$ if and only if the kernel of S_{i+1} is contained in the range of S_i.

Finally, for $S \in \mathfrak{L}$, $\alpha(S)$ will denote the dimension of the kernel of S, ker S, and $\beta(S)$ the codimension of its range.

3. The result.

THEOREM. *Suppose we have the following diagram of bounded linear operators between Banach spaces.*

$$\begin{array}{ccccccc} X_0 & \xrightarrow{S_1} & X_1 & \xrightarrow{S_2} & X_2 & \xrightarrow{S_3} & X_3 \\ {\scriptstyle A_0}\downarrow & & {\scriptstyle A_1}\downarrow & & {\scriptstyle A_2}\downarrow & & {\scriptstyle A_3}\downarrow \\ Y_0 & \xrightarrow[T_1]{} & Y_1 & \xrightarrow[T_2]{} & Y_2 & \xrightarrow[T_3]{} & Y_3 \end{array}$$

Suppose $\mathfrak{M}$ *is an ideal of* $\mathfrak{L}$ *for which* S_2, S_3, T_1, T_2 *have partial inverses modulo* $\mathfrak{M}$; $S_2 \triangle S_3$ *and* $T_1 \triangle T_2$ *belong to* $\mathfrak{M}$; *the diagram commutes modulo* $\mathfrak{M}$; A_0 *has a right inverse modulo* $\mathfrak{M}$ *and* A_3 *a left inverse modulo* $\mathfrak{M}$.

(a) *If* $S_2 S_1 \in \mathfrak{M}$ *and* A_1 *has a left inverse modulo* $\mathfrak{M}$, *then* A_2 *has a left inverse modulo* $\mathfrak{M}$.

(b) *If* $T_3 T_2 \in \mathfrak{M}$ *and* A_2 *has a right inverse modulo* $\mathfrak{M}$, *then* A_1 *has a right inverse modulo* $\mathfrak{M}$.

PROOF. (a) Suppose $S_2 S_1 \in \mathfrak{M}$ and A_1 has a left inverse modulo $\mathfrak{M}$, say A_1'. Let S_2', S_3', T_1', T_2' be, respectively, pseudo inverses modulo $\mathfrak{M}$ of S_2, S_3, T_1, T_2. Let A_0' be a right inverse modulo $\mathfrak{M}$ of A_0 and A_3' a left inverse modulo $\mathfrak{M}$ of A_3.

We exhibit a bounded linear operator $A_2'\colon Y_2 \to X_2$ such that $I_{X_2} - A_2' A_2 \in \mathfrak{M}$. In fact, let

$$A_2' = (I_{X_2} - S_2 A_1' T_2' A_2) S_3' A_3' T_3 + S_2 A_1' T_2'$$

and $M = M_1 + \cdots + M_9$, where

$$M_1 = (I_{X_2} - S_2A_1'T_2'A_2)(I_{X_2} - S_2S_2')(I_{X_2} - S_3'S_3),$$

$$M_2 = S_2A_1'(I_{Y_1} - T_1T_1')(I_{Y_1} - T_2'T_2)A_1S_2'(I_{X_2} - S_3'S_3),$$

$$M_3 = -S_2A_1'(A_1S_1 - T_1A_0)A_0'T_1'(I_{Y_1} - T_2'T_2)A_1S_2'(I_{X_2} - S_3'S_3),$$

$$M_4 = -S_2A_1'T_2'(A_2S_2 - T_2A_1)S_2'(I_{X_2} - S_3'S_3),$$

$$M_5 = (I_{X_2} - S_2A_1'T_2'A_2)S_3'A_3'(A_3S_3 - T_3A_2),$$

$$M_6 = S_2A_1'T_1(I_{Y_0} - A_0A_0')T_1'(I_{Y_1} - T_2'T_2)A_1S_2'(I_{X_2} - S_3'S_3),$$

$$M_7 = (I_{X_2} - S_2A_1'T_2'A_2)S_3'(I_{X_3} - A_3'A_3)S_3,$$

$$M_8 = S_2S_1A_0'T_1'(I_{Y_1} - T_2'T_2)A_1S_2'(I_{X_2} - S_3'S_3),$$

$$M_9 = S_2(I_{X_1} - A_1'A_1)(I_{X_1} - S_1A_0'T_1'A_1 + S_1A_0'T_1'T_2'T_2A_1)S_2'(I_{X_2} - S_3'S_3).$$

It is elementary to check that $I_{X_2} - A_2'A_2 = M$. Moreover, since $\mathfrak{M}$ is an ideal of $\mathfrak{L}$, it follows from the data that each $M_j \in \mathfrak{M}$. For example, $M_1 = (I_{X_2} - S_2A_1'T_2'A_2)(S_2 \triangle S_3) \in \mathfrak{M}$ since $S_2 \triangle S_3 \in \mathfrak{M}$. Hence $M \in \mathfrak{M}$ and (a) is proved.

(b) Suppose $T_3T_2 \in \mathfrak{M}$ and A_2 has a right inverse modulo $\mathfrak{M}$, say A_2'. Using the notation of part (a) we exhibit a bounded linear operator $A_1'\colon Y_1 \to X_1$ such that $I_{Y_1} - A_1A_1' \in \mathfrak{M}$. In fact, let

$$A_1' = S_1A_0'T_1'(I_{Y_1} - A_1S_2'A_2'T_2) + S_2'A_2'T_2$$

and $N = N_1 + \cdots + N_9$, where

$$N_1 = (I_{Y_1} - T_1T_1')(I_{Y_1} - T_2'T_2)(I_{Y_1} - A_1S_2'A_2'T_2),$$

$$N_2 = (I_{Y_1} - T_1T_1')T_2'A_2(I_{X_2} - S_2S_2')(I_{X_2} - S_3'S_3)A_2'T_2,$$

$$N_3 = -(A_1S_1 - T_1A_0)A_0'T_1'(I_{Y_1} - A_1S_2'A_2'T_2),$$

$$N_4 = (I_{Y_1} - T_1T_1')T_2'(A_2S_2 - T_2A_1)S_2'A_2'T_2,$$

$$N_5 = (I_{Y_1} - T_1T_1')T_2'A_2(I_{X_2} - S_2S_2')S_3'A_3'(A_3S_3 - T_3A_2)A_2'T_2,$$

$$N_6 = T_1(I_{Y_0} - A_0A_0')T_1'(I_{Y_1} - A_1S_2'A_2'T_2),$$

$$N_7 = (I_{Y_1} - T_1T_1')T_2'A_2(I_{X_2} - S_2S_2')S_3'(I_{X_3} - A_3'A_3)S_3A_2'T_2,$$

$$N_8 = (I_{Y_1} - T_1T_1')T_2'A_2(I_{X_2} - S_2S_2')S_3'A_3'T_3T_2,$$

$$N_9 = (I_{Y_1} - T_1T_1')T_2'(I_{Y_2} + A_2S_2S_2'S_3'A_3'T_3 - A_2S_3'A_3'T_3)(I_{Y_2} - A_2A_2')T_2.$$

As before, $I_{Y_1} - A_1A_1' = N$ and $N \in \mathfrak{M}$. So (b) is proved.

4. Applications.

4.1. We first make the following observation. Consider the situation given in the theorem and suppose, in addition, that $\mathfrak{M} = \mathfrak{FR}$. Let

$$\begin{aligned}\kappa = \operatorname{rank}(S_2 \triangle S_3) &+ \operatorname{rank}(T_1 \triangle T_2) + \operatorname{rank}(A_1 S_1 - T_1 A_0) \\ &+ \operatorname{rank}(A_2 S_2 - T_2 A_1) \\ &+ \operatorname{rank}(A_3 S_3 - T_3 A_2) + \beta(A_0) + \alpha(A_3).\end{aligned}$$

It follows from the proof of the theorem that we also have the following estimates.

(a) $$\alpha(A_2) \leqslant \kappa + \alpha(A_1) + \operatorname{rank}(S_2 S_1),$$

(b) $$\beta(A_1) \leqslant \kappa + \beta(A_2) + \operatorname{rank}(T_3 T_2).$$

To see this, note that A_3 is lsF and, in particular, has complemented kernel and range. So it has a pseudo inverse modulo 0, say A_3''. Hence, $(I_{X_3} - A_3'' A_3)$ is a projection onto the kernel of A_3 and we can take $A_3' = A_3''$ as a left inverse modulo $\mathcal{FR}$ of A_3. Moreover, $\alpha(A_3) = \operatorname{rank}(I_{X_3} - A_3' A_3)$.

Similarly, A_0 has complemented kernel and range and therefore a pseudo inverse modulo 0, say A_0''. We can take $A_0' = A_0''$ as a right inverse modulo $\mathcal{FR}$ of A_0 had hence $A_0 A_0'$ is a projection onto the range of A_0 with $\beta(A_0) = \operatorname{rank}(I_{Y_0} - A_0 A_0')$.

Since $\alpha(A_2) \leqslant \alpha(A_2' A_2) \leqslant \operatorname{rank}(I_{X_2} - A_2' A_2)$ and $\beta(A_1) \leqslant \beta(A_1 A_1') \leqslant \operatorname{rank}(I_{Y_1} - A_1 A_1')$, the result follows.

4.2. In the study of elliptic partial differential equations with boundary conditions, one is led to consider the Fredholm properties of the operator (A, B): $X \to Y \times Z$, where A is the elliptic operator, B is the boundary operator, and the spaces are chosen appropriately. It is sometimes easier to consider $B/\ker A$. This corresponds to the classical procedure of replacing an inhomogeneous partial differential equation by a homogeneous one. For this purpose, we consider the following diagram of mappings in $\mathcal{L}$, where for simplicity, the spaces are supposed to be Hilbert spaces.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker A & \xrightarrow{\subset} & X & \xrightarrow{A} & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle \left[\begin{smallmatrix}A\\B\end{smallmatrix}\right]} & & \downarrow{\scriptstyle I} & & \downarrow \\
0 & \longrightarrow & Z & \xrightarrow{\left[\begin{smallmatrix}0\\I\end{smallmatrix}\right]} & Y \oplus Z & \xrightarrow{[I\ 0]} & Y & \longrightarrow & 0
\end{array}$$

The diagram commutes, and the bottom row is exact. Hence we can apply the theorem with $\mathfrak{M} = \mathcal{FR}$. It follows that, if A has closed range, then (A, B) is lsF if and only if $B/\ker A$ is lsF. If, in addition, A is rsF, then $A \triangle 0 \in \mathcal{FR}$ and the top row is exact modulo $\mathcal{FR}$. Hence (A, B) is rsF if and only if A and $B/\ker A$ are rsF.

By applying the theorem with $\mathfrak{M} = 0$ it can be seen that the same results are true with lsF and rsF replaced by left invertible and right invertible, respectively.

4.3. The five lemma can also be used to relate operators corresponding to boundary value problems with associated sesquilinear forms. In fact, suppose A is a second order elliptic partial differential operator on a bounded domain Ω of $\mathbf{R}^n$ with smooth boundary Γ. Let B be a first order boundary operator and suppose all coefficients are smooth. Suppose $J[u, v]$ is a bounded sesquilinear form on $H^1(\bar{\Omega}) \times H^1(\bar{\Omega})$ satisfying $J[u, v] = \langle Au, v\rangle$ on $H^1(\bar{\Omega}) \times \mathring{H}^1(\bar{\Omega})$, and $J[u, v] = \langle Bu, \gamma v\rangle$ on $\ker A \times H^1(\bar{\Omega})$, where γ denotes the trace operator. Finally, let $T_J\colon H^1(\bar{\Omega}) \to H^1(\bar{\Omega})^*$ be defined by $\langle T_J u, v\rangle = J[u, v]$. Then the following diagram of bounded linear operators between Hilbert spaces commutes.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\ker A}(\bar{\Omega}) & \xrightarrow{\subset} & H^1(\bar{\Omega}) & \xrightarrow{A} & H^{-1}(\bar{\Omega}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle T_J} & & \downarrow{\scriptstyle I} & & \\
0 & \longrightarrow & H^{-1/2}(\Gamma) & \xrightarrow[\gamma^*]{} & H^1(\bar{\Omega})^* & \xrightarrow[i^*]{} & H^{-1}(\bar{\Omega}) & \longrightarrow & 0 \\
(0 & \longleftarrow & H^{1/2}(\Gamma) & \xleftarrow[\gamma]{} & H^1(\bar{\Omega}) & \xleftarrow[i]{} & \mathring{H}^1(\bar{\Omega}) & \longleftarrow & 0).
\end{array}
$$

Here, $H^1_{\ker A}(\bar{\Omega})$ denotes the kernel of A in $H^1(\bar{\Omega})$ and i is the inclusion. By consideration of the sequence in brackets, it follows that the bottom row of the diagram is exact. If A is rsF then $A \bigtriangleup 0 \in \mathcal{FR}$ and hence the top row is exact modulo $\mathcal{FR}$. It follows from the theorem that $B/\ker A$ is lsF (rsF) if and only if T_J is lsF (rsF).

5. **Extensions.** Let $\mathcal{G}$ denote the category of abelian groups and homomorphisms. Let $\mathcal{A}$ be any *additive subcategory* of $\mathcal{G}$ in the sense that it is a subcategory for which hom(A, B) is a subgroup of the group of all homomorphisms from A to B, for each $A, B \in \mathcal{A}$. Let hom $\mathcal{A}$ denote the class of all maps in $\mathcal{A}$. Then $\mathcal{M}$ is called an *ideal* of $\mathcal{A}$ if it is a nonempty proper subclass of hom $\mathcal{A}$ closed under composition with members of hom $\mathcal{A}$ and under addition.

Replacing $\mathcal{B}$ by $\mathcal{A}$ and $\mathcal{L}$ by hom $\mathcal{A}$, the definitions of left, right and pseudo inverses modulo an ideal $\mathcal{M}$, and of $S_1 \bigtriangleup S_2$, can be made as before. Since the proof of the theorem is purely algebraic, being based on the distributive property of composition over addition and on the commutativity of addition in $\mathcal{A}$, the thoerem remains valid.

Examples of additive subcategories of $\mathcal{G}$ are $\mathcal{G}$ itself, vector spaces and linear operators, locally convex spaces and continuous linear operators, abelian topological groups and continuous homomorphisms, and so on.

Consider the category $\mathcal{V}$ of vector spaces and linear operators. If we assume the axiom of choice, it follows that every map in $\mathcal{V}$ has a partial inverse modulo 0. Moreover, such a map is one-to-one (onto) if and only if it has a left (right) inverse modulo 0. Hence our result recovers the standard five lemma applied to $\mathcal{V}$.

Finally, it should be noted that my Ph.D. thesis, and indeed a forthcoming paper, are concerned with the Fredholm properties of operators associated with elliptic partial differential equations with mixed boundary conditions. The problems therein are repeatedly reduced to equivalent problems, until solved, and by application of the five lemma the proofs are greatly simplified.

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