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QUASISIMILARITY DOES NOT PRESERVE THE HYPERLATTICE

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ABSTRACT. Two quasisimilar nilpotent Hilbert space operators of order three can have nonisomorphic hyperinvariant subspace lattices.

1. **Introduction**. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all (bounded linear) operators acting on the complex separable Hilbert space $\mathcal{H}$. If $X \in \mathcal{L}(\mathcal{H})$ and $\operatorname{Ker} X = \operatorname{Ker} X^* = \{0\}$, then X is called a *quasiaffinity*. If $A, B \in \mathcal{L}(\mathcal{H})$ and there exist X, Y such that $AX = XB$ and $YA = BY$, then A and B are said to be *quasi-similar*. It is known that if A and B are quasisimilar operators, and A has a non-trivial hyperinvariant (i.e., invariant under the commutant $\mathcal{A}'(A)$ of A) subspace, then so does B (see [4], [7], [9]). Furthermore, if A is normal then quasisimilarity induces an injection from Hyperlat A into Hyperlat B (see [8]; *Hyperlattice* is an abbreviation for *lattice of hyperinvariant subspaces*), so one could expect that quasisimilar operators always have isomorphic hyperlattices. An example will show that this is not necessarily true, even for very simple operators.

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2. **The hyperlattice of certain nilpotent operators.**

LEMMA 1. *Let Q be a nilpotent operator of order three* ($Q^3 = 0$). *Then* $\operatorname{Ker} Q^2$ (($\operatorname{Ran} Q^2)^-$, *resp.*) *is a maximal* (*minimal, resp.*) *hyperinvariant subspace of Q.*

PROOF. Let

$$(1) \qquad Q = \begin{bmatrix} 0 & Q_{12} & Q_{13} \\ 0 & 0 & Q_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

be the matrix of Q with respect to the orthogonal direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where $\mathcal{H}_1 = \operatorname{Ker} Q$, $\mathcal{H}_2 = \operatorname{Ker} Q^2 \ominus \operatorname{Ker} Q$ and $\mathcal{H}_3 = \mathcal{H} \ominus \operatorname{Ker} Q^2$. Then [3] Q_{12} and Q_{23} are injective operators and therefore their adjoints have dense ranges.

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A straightforward computation shows that the commutant of Q consists of all those operators $A \in \mathcal{L}(\mathcal{H})$ of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \tag{2}$$

such that $A_{11}Q_{12} = Q_{12}A_{22}$, $A_{22}Q_{23} = Q_{23}A_{33}$ and $A_{11}Q_{13} + A_{12}Q_{23} = Q_{12}A_{23} + Q_{13}A_{33}$ (A_{13} can be arbitrarily chosen).

Let $\mathfrak{M} \in \text{Hyperlat}\, Q$ and assume that $\mathfrak{M}$ is not contained in $\text{Ker}\, Q^2$; then there exists a vector (e, f, g) in $\mathfrak{M}$ ($e \in \mathcal{H}_1, f \in \mathcal{H}_2, g \in \mathcal{H}_3$) with $g \neq 0$. Let A be as in (2) with $A_{jk} = 0$ for $(j, k) \neq (1, 3)$; then the hyperinvariance of $\mathfrak{M}$ implies that $A(e, f, g) = (A_{13}g, 0, 0) \in \mathfrak{M}$. Since A_{13} can be arbitrarily chosen, we conclude that $\mathcal{H}_1 \subset \mathfrak{M}$. Hence $(0, f, g) \in \mathfrak{M}$.

Since $\text{Ran}\, Q_{23}^*$ is dense, there exists an $f_0 \in \mathcal{H}_2$ such that $\langle Q_{23}^* f_0, g\rangle = 1$. Let f_2 be an element of $\mathcal{H}_2$ and define $B_{23} = f_2 \otimes Q_{23}^* f_0$, $B_{12} = Q_{12}f_2 \otimes f_0$ (where $x \otimes y$ denotes the operator defined by $x \otimes y(z) = \langle z, y\rangle x$) and $B_{jk} = 0$ for all $(j, k) \neq (1, 2)$ or $(2, 3)$. It is easily seen that $B = (B_{jk}) \in \mathcal{A}'(Q)$ and therefore $B(0, f, g) = (B_{12}f, B_{23}g, 0) = (B_{12}f, f_2, 0) \in \mathfrak{M}$, whence it readily follows that $\mathcal{H}_1 \oplus \mathcal{H}_2 = \text{Ker}\, Q^2 \subset \mathfrak{M}$. Hence $(0, 0, g) \in \mathfrak{M}$.

Now use the fact that $\text{Ran}(Q_{12}Q_{23})^*$ is dense in order to obtain an $e_0 \in \mathcal{H}_1$ such that $\langle (Q_{12}Q_{23})^* e_0, g\rangle = 1$. Let g_3 be an element of $\mathcal{H}_3$ and define $C_{33} = g_3 \otimes (Q_{12}Q_{23})^* e_0$, $C_{22} = Q_{23}g_3 \otimes Q_{12}^* e_0$, $C_{11} = Q_{12}Q_{23}g_3 \otimes e_0$, $C_{12} = Q_{13}g_3 \otimes Q_{12}^* e_0$, $C_{23} = Q_{23}g_3 \otimes Q_{13}^* e_0$ and $C_{jk} = 0$ for all $(j, k) \neq (1, 1), (1, 2), (2, 2), (2, 3)$ or $(3, 3)$. Then $C = (C_{jk}) \in \mathcal{A}'(Q)$ and therefore $C(0, 0, g) = (0, C_{23}g, C_{33}g) = (0, C_{23}g, g_3) \in \mathfrak{M}$, whence we conclude that $\mathfrak{M} = \mathcal{H}$.

The same arguments applied to Q^* shows that $\text{Ker}\, Q^{*2}$ is a maximal hyperinvariant subspace of Q^* and therefore $(\text{Ker}\, Q^{*2})^\perp = (\text{Ran}\, Q^2)^-$ is a minimal hyperinvariant subspace of Q. □

COROLLARY 2. *If* $\mathfrak{M} \in \text{Hyperlat}\, Q$, $Q^3 = 0$ *and* $\mathfrak{M} \neq \{0\}, \mathcal{H}$, *then*

$$(\text{Ran}\, Q^2)^- \subset \mathfrak{M} \subset \text{Ker}\, Q^2.$$

Let $q_k \in \mathcal{L}(\mathbf{C}^k)$ be the nilpotent operator defined by $q_k e_1 = 0$, $q_k e_j = e_{j-1}$ for $j = 2, 3, \ldots, k$, with respect to the canonical ONB $\{e_j\}_{j=1}^k$ of $\mathbf{C}^k$ and let $q_k(\alpha_k)$ be the orthogonal direct sum of α_k copies of q_k acting in the usual fashion on the orthogonal direct sum of α_k copies of $\mathbf{C}^k$. An operator $J \in \mathcal{L}(\mathcal{H})$ is a *Jordan operator* [2] if it can be written as $J = \bigoplus_{k=1}^n q_k(\alpha_k)$ with respect to a suitable decomposition

$$\mathcal{H} = \bigoplus_{k=1}^{n} \left(\bigoplus_{j=1}^{\alpha_k} \mathbf{C}^k_{(j)} \right)$$

of $\mathcal{H}$.

LEMMA 3. *Let* $T \in \mathcal{L}(\mathcal{H})$, *let* $E \in \mathcal{A}'(T)$ *be an idempotent operator and let*

$\mathfrak{M} \in$ Hyperlat T; *then* $E\mathfrak{M} = \mathfrak{M} \cap$ Ran $E \in$ Hyperlat T_E, *where* T_E *denotes the restriction of* T *to* Ran E.

Let $\mathfrak{N}$ *be a subspace and let* $\{E_\nu\}_{\nu\in\Gamma}$ *be a uniformly bounded chain of idempotents in* $\mathfrak{A}'(T)$ (*if* $\nu < \mu$ *in* Γ, *then* Ran $E_\nu \subset$ Ran E_μ) *such that* $E_\nu \to I$ (*strongly*). *If* $\mathfrak{N}_\nu = E_\nu \mathfrak{N}^-$, *then* $\mathfrak{N} \in$ Hyperlat T *if and only if* $\mathfrak{N}_\nu \in$ Hyperlat T_{E_ν} *for all* $\nu \in \Gamma$.

PROOF. Since $E = E^2 \in \mathfrak{A}'(T)$, for every $A \in \mathfrak{A}'(T)$ the restriction A_E of EAE to Ran E belongs to $\mathfrak{A}'(T_E)$.

Conversely, if $B_0 \in \mathfrak{A}'(T_E)$, then B_0 can be extended to an element B of $\mathfrak{A}'(T)$ defined by $Bx = B_0x$ if $x \in$ Ran E and $Bx = 0$ if $x \in$ Ran$(I - E)$. It readily follows that $(E\mathfrak{M})^- \in$ Hyperlat T_E. On the other hand, it is easily seen that $E\mathfrak{M} = \mathfrak{M} \cap$ Ran E and therefore $E\mathfrak{M}$ is closed. This proves the first statement.

For the second one it is enough to observe that $E_\nu A E_\nu \to A$ for every $A \in \mathfrak{L}(\mathfrak{H})$. Hence $A \in \mathfrak{A}'(T)$ if and only if $E_\nu A E_\nu|$Ran $E_\nu \in \mathfrak{A}'(T_{E_\nu})$ for all ν, whence the result follows. □

PROPOSITION 4. *If* $J = \bigoplus_{k=1}^n q_k(\alpha_k)$ *is a Jordan operator, then* Hyperlat J *is the lattice generated by* $\{\text{Ker } J^k, \text{Ran } J^k\}$ ($k = 0, 1, 2, \ldots, n$) *and it is order-isomorphic with* Hyperlat $\bigoplus_{h=1}^m q_{k_h}$, *where* $\{k_h\}_{h=1}^m$ *is the subset of* $\{1, 2, \ldots, n\}$ *corresponding to those indices such that* $\alpha_k \neq 0$.

PROOF. Let P_N be the orthogonal projection of $\mathfrak{H}$ onto

$$\mathfrak{H}_N = \bigoplus_{k=1}^{n} \left(\bigoplus_{j=1}^{\min[\alpha_k, N]} \mathbf{C}^k_{(j)} \right)$$

and let J_N be the restriction of J to $\mathfrak{H}_N$.

Then Hyperlat J_N is generated by $\{\text{Ker } J_N^k, \text{Ran } J_N^k\colon k = 0, 1, 2, \ldots, n\}$ and it is order-isomorphic with Hyperlat $\bigoplus_{h=1}^m q_{k_h}$ (see [1], [5]). Now the result follows from the analysis of the hyperlattice of a finite dimensional operator carried out in the above references and Lemma 3. □

3. **The example**. According to [2, Theorem 1] (or [10]), every nilpotent operator in $\mathfrak{L}(\mathfrak{H})$ is quasisimilar to a Jordan operator.

Let $\mathfrak{H} = \mathbf{C} \oplus l_1^2 \oplus l_2^2 \oplus l_3^2$ and let $T \in \mathfrak{L}(\mathfrak{H})$ be the operator defined by the matrix

$$T = \begin{bmatrix} 0 & 0 & 0 & T_{13a} \\ 0 & 0 & T_{12b} & 0 \\ 0 & 0 & 0 & T_{23} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

where $T_{13a}(\{c_n\}) = \sum_{n=1}^\infty c_n/n$ ($T_{13a}\colon l_3^2 \to \mathbf{C}$), $T_{12b}(\{c_n\}) = \{c_n\}$ ($T_{12b}; l_2^2 \to l_1^2$) and $T_{23}(\{c_n\}) = \{c_n/n^2\}$ ($T_{23}\colon l_3^2 \to l_2^2$).

PROPOSITION 5. Hyperlat T *is the chain of five elements* $\{0\} \subset (\text{Ran } T^2)^- \subset$ Ker $T \subset (\text{Ran } T)^- =$ Ker $T^2 \subset \mathfrak{H}$.

COROLLARY 6. *There exist two quasisimilar nilpotent operators T and J of order three such that* Hyperlat T *and* Hyperlat J *do not contain the same (finite) number of elements. In particular, these lattices are not order-isomorphic.*

PROOF. Choose T given by (3). It is easily seen that $T^3 = 0$, so that ([2, Theorem 1]; see also [6], [10]) there exists a Jordan operator J quasisimilar to T.

By Proposition 5, Hyperlat T has five elements, but, by Proposition 4 and the results of [1], [5], Hyperlat J can only have four, six or eight elements. □

PROOF OF PROPOSITION 5. $(\operatorname{Ran} T^2)^- = (\operatorname{Ran} T_{12b}T_{23})^- = l_1^2$, $\operatorname{Ker} T = \mathbf{C} \oplus l_1^2$, $\operatorname{Ker} T^2 = \mathbf{C} \oplus l_1^2 \oplus l_2^2$ and $\operatorname{Ran} T = \{T_{13a}g, T_{12b}f, T_{23}g, 0: f \in l_2^2, g \in l_3^2\}$. Clearly, $l_1^2 \subset \operatorname{Ran} T$. Let $f_1 = \{c_1, c_2, \ldots, c_N, 0, 0, \ldots\} \in l_2^2$, let $\lambda \in \mathbf{C}$ and choose $g_n = \{c_1, 4c_2, \ldots, N^2c_N, 0, 0, \ldots, 0, d_n, 0, \ldots\} \in l_3^2$, where $d_n = n(\lambda - \sum_{j=1}^N jc_j)$, $n = 1, 2, \ldots$; then

$$\|T(0,0,0,g_n) - (\lambda, 0, f_1, 0)\| = |d_n/n^2|$$
$$\leqslant (1/n)\left(|\lambda| + \sum_{j=1}^{N} j|c_j|\right) \to 0 \qquad (n \to \infty).$$

Therefore $\mathbf{C} \oplus l_2^2 \subset (\operatorname{Ran} T)^-$, i.e. $(\operatorname{Ran} T)^- = \mathbf{C} \oplus l_1^2 \oplus l_2^2 = \operatorname{Ker} T^2$.

Let $\mathfrak{M} \in$ Hyperlat T, $\mathfrak{M} \neq \{0\}, \mathcal{H}$. By Corollary 2, $(\operatorname{Ran} T^2)^- \subset \mathfrak{M} \subset \operatorname{Ker} T^2$. Assume that $(\lambda, 0, f, 0) \in \mathfrak{M}$ for some $f \in l_2^2, f \neq 0$, and let $\{f_m\}_{m=1}^\infty$ be the canonical ONB of l_2^2. Let p be the first nonzero coordinate of f, i.e., $\langle f, f_p\rangle \neq 0$ and define

$$A_m = \begin{bmatrix} 0 & 0 & A_{12a}(m) & 0 \\ 0 & A_{11b}(m) & 0 & 0 \\ 0 & 0 & A_{22}(m) & 0 \\ 0 & 0 & 0 & A_{33}(m) \end{bmatrix}$$

where $A_{11b}(m) = A_{22}(m) = f_m \otimes f_p$, $A_{33}(m) = (m/p)^2 f_m \otimes f_p$ and $A_{12a}(m)(\{c_n\}) = mc_p$, $m = 1, 2, 3, \ldots$. It can be easily checked that $\{A_m\}_{m=1}^\infty \subset \mathcal{Q}'(T)$ and that $A_m(\lambda, 0, f, 0) = (A_{12a}(m)f, 0, A_{22}(m)f, 0) = \langle f, f_p\rangle (m, 0, f_m, 0) \in \mathfrak{M}$.

Hence, $(1/m\langle f, f_p\rangle)A_m(\lambda, 0, f, 0) \to (1, 0, 0, 0)$ $(m \to \infty)$ and therefore $\mathbf{C} \subset \mathfrak{M}$. A fortiori, $(0, 0, f_m, 0) = (m, 0, f_m, 0) - (m, 0, 0, 0) \in \mathfrak{M}$ and therefore $\mathfrak{M} = \mathbf{C} \oplus l_1^2 \oplus l_2^2 = \operatorname{Ker} T^2$.

On the other hand, if $(\lambda, 0, f, 0) \in \mathfrak{M}$ implies $f = 0$, then it is easy to see that either $\mathfrak{M} = l_1^2 = (\operatorname{Ran} T)^-$ (and $\lambda \equiv 0$) or $\mathfrak{M} = \mathbf{C} \oplus l_1^2 = \operatorname{Ker} T$. □

REFERENCES

1. T. Andô and T. Sekiguchi, *Hyperinvariant subspaces of a nilpotent operator* (preprint).

2. C. Apostol, R. G. Douglas and C. Foiaş, *Quasisimilar models for nilpotent operators* (to appear).

3. J. Barría and D. A. Herrero, *Closure of similarity orbits of nilpotent operators*. II (preprint).

4. P. A. Fillmore, *Notes in operator theory*, Mathematical Studies No. 30, Van Nostrand, Princeton, N.J., 1970.

5. P. A. Fillmore, D. A. Herrero and W. E. Longstaff, *The hyperinvariant subspace lattice of a linear transformation*, Linear Algebra and Appl. (to appear).

6. C. Foiaş and C. Pearcy, *A model for quasinilpotent operators*, Michigan Math. J. **21** (1974), 399–404.

7. T. B. Hoover, *Quasisimilarity of operators*, Illinois J. Math. **16** (1972), 678–686. MR **47** #866.

8. ______, *Hyperinvariant subspaces for n-normal operators*, Acta Sci. Math. (Szeged) **32** (1971), 109–119.

9. B. Sz.-Nagy and C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson et Cie., Paris, Akademiai Kiadó, Budapest, 1967. MR **37** #778.

10. L. Williams, *Similarity invariants for a class of nilpotent operators*, Acta Sci. Math. (Szeged) (to appear).

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