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A CLASS OF SINGULAR NEUTRAL-DIFFERENTIAL SYSTEMS[1]

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ABSTRACT. Noetherian operator theory is used to prove an existence theorem for a singular functional-differential system. An analogue of the standard existence and uniqueness results at an ordinary point follows as a corollary.

Functional-differential equations of the form

$$y'(z) = A(z)y(z) + B(z)y(\alpha z) \tag{1}$$

have been studied by T. Kato and J. B. McLeod [6] and others. Recently Ll. G. Chambers [1] has obtained sufficient conditions for existence of holomorphic solutions of the scalar equation with deviating arguments and constant coefficients of the form

$$y'(z) = \sum_{n=0}^{\infty} a_n y(\mu^n z), \tag{2}$$

where μ is a constant with $|\mu| < 1$, and V. P. Skripnik [8] has studied stability of a corresponding nonlinear equation.

A singular equation analogous to (1) has also received some attention, particularly in the Soviet literature; see, for example, the references in [2] or [4]. We obtain here existence results for a singular neutral-differential system with deviating arguments similar to (2). The principal result is a generalization of the classical Lettenmeyer theorem [5].

Let B_1 and B_2 be Banach spaces and let T: $B_1 \to B_2$ be a continuous linear operator with domain B_1. Denote the conjugate of T by T^*.

DEFINITION 1. The operator T is called normally solvable if the set $T(B_1)$ is closed in B_2.

DEFINITION 2. The d-characteristic of the operator T is the ordered pair $(\alpha(T), \beta(T))$, where $\alpha(T) = \dim(\ker T)$ and $\beta(T) = \dim(B_2/T(B_1))$.

DEFINITION 3. The index of the operator T is the number $\beta(T) - \alpha(T)$, denoted by $\operatorname{ind}(T)$.

DEFINITION 4. The operator T is called a Noetherian operator if T is normally solvable and has a finite d-characteristic, i.e., both $\alpha(T)$ and $\beta(T)$ are finite.

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REMARK. If $\beta(T) < \infty$, then $\beta(T) = \dim(\ker T^*)$.

The following theorem is due to I. C. Gohberg and M. G. Kreĭn [3].

THEOREM 1. *Let T: $B_1 \to B_2$ be a continuous Noetherian operator, and let T_1: $B_1 \to B_2$ be a compact operator. Then $T_2 = T + T_1$ is a Noetherian operator from B_1 into B_2 with* $\operatorname{ind}(T_2) = \operatorname{ind}(T)$.

Let $r > 0$, let $G_r = \{z: |z| < r\}$ and define $A_p(G_r)$ to be the Banach space of functions $v(z)$ holomorphic in G_r and p times continuously differentiable on $\bar{G}_r$ with norm

$$\|v(z)\|_p = \max_{z \in \bar{G}_r} |v^{(i)}(z)|, \qquad 0 \leqslant i \leqslant p.$$

Let $A_{p,n}(G_r)$ be the Banach space of n-vector functions $y(z) = (y^1(z), \ldots, y^n(z))^T$, where $y^k(z) \in A_p(G_r)$, $1 \leqslant k \leqslant n$, with norm

$$\|y(z)\|_p^n = \max \|y^k(z)\|_p, \qquad 1 \leqslant k \leqslant n.$$

For a matrix $A(z) = (a_{ij}(z))$, with $a_{ij}(z) \in A_0(G_r)$, define

$$\|A(z)\| = \max \|a_{ij}(z)\|_0, \qquad 1 \leqslant i, j \leqslant n.$$

Note that $\|A(z)y(z)\|_0^n = \|A(z)\| \, \|y(z)\|_0^n$.

Consider the differential equation

$$A(z)y'(z) + \sum_{i=1}^{\infty} B_i(z)y(\alpha_i z) + \sum_{i=1}^{\infty} C_i(z)y'(\beta_i z) = 0, \tag{3}$$

where $A(z)$, $B_i(z)$, $C_i(z)$ are $n \times n$ matrices with elements in $A_0(G_r)$, $\sum_{i=1}^{\infty} \|B_i(z)\| < \infty$, $\sum_{i=1}^{\infty} \|C_i(z)\| < \infty$, and $\{\alpha_i\}$ and $\{\beta_i\}$ are sequences of complex constants with $0 < |\alpha_i| \leqslant 1$, $0 < |\beta_i| < 1$.

THEOREM 2. *Let s be the number of zeros, counted algebraically, of* $\det A(z)$ *in G_r. Then the system* (3) *has at least $n - s$ linearly independent solutions holomorphic in G_r.*

PROOF. Consider the operators $l^{(1)}$ and $l^{(2)}$ defined by

$$l^{(1)}y(z) = B(z)y(\alpha z), \qquad |\alpha| \leqslant 1,$$

$$l^{(2)}y(z) = C(z)y'(\beta z), \qquad |\beta| < 1;$$

$l^{(1)}$ and $l^{(2)}$ each map $A_{1,n}(G_r)$ into $A_{0,n}(G_r)$. Let Q be the family of n-vectors $h(z) \in A_1$, $n(G_r)$ such that $\|h\|_1^n \leqslant C < \infty$ for all $h \in Q$. Let $\hat{h}(z) = h(\alpha z)$. For all $h(z) = (h_1(z), \ldots, h_n(z)) \in Q$, each $j = 1, \ldots, n$, and $z \in \bar{G}_r$,

$$|\hat{h}_j(z)| = |h_j(\alpha z)| \leqslant \sup_{z \in \bar{G}_r} |h_j(z)| \leqslant C \quad \text{and}$$

$$|\hat{h}_j'(z)| = |\alpha h_j'(\alpha z)| \leqslant \sup_{z \in \bar{G}_r} |h_j'(z)| \leqslant C,$$

by the maximum modulus theorem. Hence the family $\{\hat{h}_j(z)\}_{h \in Q}$ is uniformly bounded and equicontinous on $\bar{G}_r$. From Ascoli's lemma, it is easy to see that

$\{\hat{h}(z)\}_{h\in Q}$ contains a sequence convergent in $A_{0,n}(G_r)$. Thus $l^{(1)}$: $A_{1,n}(G_r) \to A_{0,n}(G_r)$ is a compact operator.

Now set $\tilde{h}(z) = h(\beta z)$. By Montel's theorem, $\{h_j'(z)\}_{h\in Q}$ is a normal family for $|z| < r$, hence $\{\tilde{h}_j'(z)\}_{h\in Q}$ is a normal family for $|z| < r/|\beta|$. This means that $\{\tilde{h}_j'(z)\}_{h\in Q}$ contains a sequence almost uniformly convergent on $|z| < r/|\beta|$. $\{z\colon |z| \leqslant r\}$ is a compact subset of $\{z\colon |z| < r/|\beta|\}$, so $\{\tilde{h}'(z)\}_{z\in Q}$ clearly contains a sequence convergent in $A_{0,n}(G_r)$. Thus $l^{(2)}$: $A_{1,n}(G_r) \to A_{0,n}(G_r)$ is a compact operator.

We have now shown that the operators $L_k^{(1)}$, $L_k^{(2)}$: $A_{1,n}(G_r) \to A_{0,n}(G_r)$ defined by

$$L_k^{(1)} y(z) = \sum_{i=1}^{k} B_i(z) y(\alpha_i z) \quad \text{and}$$

$$L_k^{(2)} y(z) = \sum_{i=1}^{k} C_i(z) y'(\beta_i z)$$

are compact.

Now consider

$$L^{(1)} y(z) = \sum_{i=1}^{\infty} B_i(z) y(\alpha_i z) \quad \text{and} \quad L^{(2)} y(z) = \sum_{i=1}^{\infty} C_i(z) y'(\beta_i z).$$

$$\|L^{(1)} y(z)\|_0^n \leqslant \sum_{i=1}^{\infty} \|B_i(z)\| \|y(\alpha_i z)\|_0^n \leqslant \sum_{i=1}^{\infty} \|B_i(z)\| \|y(z)\|_1^n.$$

Since $\sum_{i=1}^{\infty} \|B_i(z)\| < \infty$, $L_k^{(1)} \to L^{(1)}$ uniformly in $A_{1,n}(G_r)$. $\{L_k^{(1)}\}$ is a sequence of compact operators, hence $L^{(1)}$ is compact. Similarly, $L^{(2)}$ is compact.

Korobeĭnik [7] has shown that the operator $L^{(0)}$: $A_{1,n}(G_r) \to A_{0,n}(G_r)$ defined by $L^{(0)}y(z) = A(z)y'(z)$ is Noetherian with index $s - n$, where s is the number of zeros, counted algebraically, of $\det A(z)$ in G_r. By Theorem 1, the operator $L = L^{(0)} + L^{(1)} + L^{(2)}$ is Noetherian with index $s - n$. But $\beta(L) \geqslant 0$, and thus $\dim(\ker L) = \alpha \geqslant n - s$. This completes the proof.

If we assume that $A(0)$ is nonsingular, we obtain the following corollary, which yields an analogue of the standard existence results at an ordinary point, from simple uniqueness considerations together with the fact that the space of functions holomorphic at zero can be represented as the limit of the spaces $A_1(G_r)$ as r tends to zero:

COROLLARY. *With notation as in* (2), *the differential equation*

$$A(z) y'(z) = \sum_{i=1}^{\infty} B_i(z) y(\alpha_i z) + \sum_{i=1}^{\infty} C_i(z) y'(\beta_i z)$$

has exactly n linearly independent solutions holomorphic at $z = 0$, *provided that* $A(0)$ *is nonsingular*.

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