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PEAK POINTS, BARRIERS AND PSEUDOCONVEX BOUNDARY POINTS[1]

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ABSTRACT. Let x be a smooth boundary point of a domain in $\mathbf{C}^n$. It is shown that x is a limit of strictly pseudoconvex boundary points whenever there is a "plurisubharmonic barrier" for x.

1. **Introduction.** Let D be an open subset of $\mathbf{C}^n$ and, as usual, let

$$A(D) = \left\{ f \in C(\bar{D}) | f \text{ is holomorphic on } D \right\}.$$

If D is bounded then $A(D)$ is a uniform algebra on $\bar{D}$ and in this case we let $S(D)$ denote the Shilov boundary of $A(D)$, a subset of ∂D.

The problem of characterizing $S(D)$ for certain pseudoconvex domains was discussed by Bremermann in [1, Theorems 6.8 and 6.9]. Related results were obtained by Rossi in [11]. Most recently Pflug has used the results of Kohn in [6] to show that $S(D)$ contains the closure of the set of strictly pseudoconvex boundary points of D when D is a psuedoconvex domain with C^∞ boundary (see Folgerung 5 in [10]); Pflug refers to [3] for the reverse inclusion, but this reference does not seem to be widely available. Debiard and Gaveau have shown that $S(D)$ is contained in the closure of the strictly pseudoconvex boundary points when D has the form $\{z | V(z) < 0\}$ for some C^3 function V defined near $\bar{D}$ which is plurisubharmonic on D and satisfies $dV \neq 0$ on ∂D. They follow earlier work of Malliavin [7]–[9] and E. M. Stein [12] in applying probabilistic potential theory to study boundary behavior in $\mathbf{C}^n$ via suitable Kählerian metrics (see [2]).

In this note we prove that $S(D)$ is contained in the closure of the strictly pseudoconvex boundary points of D when D is any bounded open subset of $\mathbf{C}^n$ with C^2 boundary. The same kind of elementary geometric considerations which Rossi utilized in [11] can be used to obtain this sharper result. (Much the same result is part of a recent announcement of Hakim and Sibony [4].) We actually show that a boundary point which has a "plurisubharmonic barrier" is a limit of strictly pseudoconvex boundary points (Corollary 1), from which the partial characterization of $S(D)$ follows at once.

2. **Preliminaries.** Let D be an open subset of $\mathbf{C}^n$. A defining function for D on a set $\Omega \subseteq \mathbf{C}^n$ is a real-valued function ϕ on Ω such that $D \cap \Omega = \{\phi < 0\}$.

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If $\alpha \in \partial D$, α is a C^2 boundary point of D if there is a neighborhood Ω of α and a C^2 defining function ϕ for D on Ω with $(d\phi)_\alpha \neq 0$. Given such an α, we may choose coordinates $z_j = x_j + iy_j$ for $\mathbf{C}^n$ so that $\alpha = 0$ and $(d\phi)_\alpha = (dx_1)_\alpha$, whence the tangent space to ∂D at α is $\{x_1 = 0\}$. If we make the identification $\mathbf{C}^n = \mathbf{R} \times (\mathbf{R} \times \mathbf{C}^{n-1})$ by $z = (x_1, (y_1, (z_2, \ldots, z_n)))$, then the implicit function theorem enables us to choose a $\delta > 0$, an open neighborhood U of 0 in $\mathbf{R} \times \mathbf{C}^{n-1}$, and a real-valued function $\psi \in C^2(U)$ with the property that

$$D \cap \left[(-\delta, \delta) \times U\right] = \left\{(x, t) \in (-\delta, \delta) \times U \mid x < \psi(t)\right\}.$$

Since $x - \psi(t)$ is a defining function for D on $(-\delta, \delta) \times U$ whose restriction to the complex tangent space to ∂D at α is $-\psi(0, (z_2, \ldots, z_n))$, α will be a strictly pseudoconvex boundary point of D precisely if ψ is strictly superharmonic at 0 on every complex line through 0 in U.

Suppose K is a compact smoothly bounded subset of $\mathbf{R}^n$, B is a closed ball, $B \supset K$, and $\alpha \in \partial B \cap \partial K$; then of course α is a strictly convex boundary point of K. The following proposition (proved by Rossi in [11]) is an intuitively obvious complex analogue of this result, and provides us with a criterion for recognizing strictly pseudoconvex boundary points.

Proposition. *Let D be an open subset of $\mathbf{C}^n$, α a C^2 boundary point of D, Ω an open neighborhood of α, $v \in C^2_{\mathbf{R}}(\Omega)$, v strictly plurisubharmonic. Suppose that $v(\alpha) = 0$ and that $v < 0$ on $D \cap \Omega$. Then α is a strictly pseudoconvex boundary point of D.*

Proof. We may assume that $(dv)_\alpha \neq 0$; for if $(dv)_\alpha = 0$ then one can replace v by $v + \varepsilon\phi$, where ϕ is any defining function for D near α with $(d\phi)_\alpha \neq 0$ and ε is a corresponding small positive number. Choose coordinates for $\mathbf{C}^n$ so that $\alpha = 0$ and $(dv)_\alpha = (dx_1)_\alpha$. Then there is an open set V in $\mathbf{C}^n$ with $\alpha \in V \subseteq \Omega$, and an open neighborhood U of 0 in $\mathbf{R} \times \mathbf{C}^{n-1}$ with functions ψ, $u \in C^2_{\mathbf{R}}(U)$, such that

$$D \cap V = \left\{(x, t) \in \mathbf{R} \times U \mid x < \psi(t)\right\};$$

$$\{v < 0\} \cap V = \left\{(x, t) \in \mathbf{R} \times U \mid x < u(t)\right\};$$

$$\left\{(\psi(t), t) \mid t \in U\right\} \subseteq V.$$

Note that $\psi(0) = u(0) = 0$ since $\alpha \in \partial D \cap \partial(\{v < 0\})$. Furthermore, $\psi \leqslant u$. [Let $t \in U$. Then for any small positive ε, $(\psi(t) - \varepsilon, t) \in D \cap V$. Since $V \subseteq \Omega$, the hypotheses about v imply $(\psi(t) - \varepsilon, t) \in \{v < 0\} \cap V$. Thus for all small positive ε, $\psi(t) - \varepsilon < u(t)$, so $\psi(t) \leqslant u(t)$.] So $\psi - u$ has a maximum at 0.

Now v is strictly plurisubharmonic and $x - u(t)$ is a defining function for $\{v < 0\}$ near 0, so u must be strictly superharmonic at 0 on complex lines through 0 in U. If ζ parametrizes such a line, then $u_{\zeta\bar{\zeta}}(0) < 0$. But $\psi - u$ has a maximum at 0, so $(\psi - u)_{\zeta\bar{\zeta}}(0) \leqslant 0$, whence $\psi_{\zeta\bar{\zeta}}(0) < 0$. Thus ψ is also strictly superharmonic at 0 on complex lines through 0 in U, from which it follows that α is a strictly pseudoconvex boundary point of Ω.

3. **Existence of strictly pseudoconvex boundary points.** We are now ready to prove a fairly general result about the existence of strictly pseudoconvex boundary points. This result has a number of direct consequences, including the characterization of the Shilov boundary of $A(D)$ mentioned in the Introduction.

THEOREM. *Let U_1, U be bounded open subsets of $\mathbf{R} \times \mathbf{C}^{n-1}$ with $\overline{U} \subseteq U_1$. Let $\psi \in C^2(U_1)$, let $\varepsilon_1 > \varepsilon > 0$, and let D, V denote the following subsets of $\mathbf{C}^n$:*

$$D = \{(x, t) \in \mathbf{R} \times U_1 | \psi(t) - \varepsilon_1 < x < \psi(t)\};$$

$$V = \{(x, t) \in \mathbf{R} \times U | \psi(t) - \varepsilon < x < \psi(t) + \varepsilon\}.$$

Suppose there is a plurisubharmonic function $u \in C^2_{\mathbf{R}}(D)$ for which $\sup_{D \setminus V} u < \sup_D u$. Then V contains a strictly pseudoconvex boundary point of D. (Note that $\partial D \cap V = \{(x, t) \in \mathbf{R} \times U | x = \psi(t)\}$ contains only C^2 boundary points of D.)

PROOF. We may assume that u is strictly plurisubharmonic on D, for u may be replaced by $u + \delta \sum_{j=1}^n |z_j|^2$ if δ is a sufficiently small positive number. Let $K = \{(x, t) \in \mathbf{R} \times \overline{U} | x = \psi(t)\}$. For $0 < s < \varepsilon_1$ define

$$D_s = (s, 0) + D = \{(x, t) \in \mathbf{R} \times U_1 | s + \psi(t) - \varepsilon_1 < x < s + \psi(t)\},$$

and note that each D_s is a neighborhood of K in $\mathbf{C}^n$. Define u_s on D_s by

$$u_s(y, t) = u(y - s, t), \qquad (y, t) \in D_s,$$

and let $m(s) = \max_K u_s$ for $0 < s < \varepsilon_1$. Notice that m is continuous on $(0, \varepsilon_1)$, that $\sup m = \sup u$, and that $m(s) \leqslant \sup_{D \setminus V} u$ when $\varepsilon \leqslant s < \varepsilon_1$. Choose c with $\sup_{D \setminus V} u < c < \sup_D u$, and let

$$r = \max\{x \in (0, \varepsilon) | m(s) \geqslant c\}.$$

Then $m(r) = c$, and $u_r < c$ on $D_r \cap D$. Choose $\alpha \in K$ so that $u_r(\alpha) = c$. It is easily seen that $\alpha \in \partial D \cap V$. Observe that $u_r - c$ is strictly plurisubharmonic on D_r, that $(u_r - c)(\alpha) = 0$, and that $u_r - c < 0$ on $D_r \cap D$. By the above proposition, α is a strictly pseudoconvex boundary point of D; since $\alpha \in V$, this completes the proof of the Theorem.

REMARK. The assumption that u is smooth in the Theorem and in Corollary 1 below is unnecessary. One need only approximate the given plurisubharmonic function by a smooth one defined on a slightly smaller domain in the usual way (as is done, e.g., for subharmonic functions in Theorem 1.6.11 in [5]), and observe that the boundary of the smaller domain may be taken to be a translate of the boundary of the original domain near the boundary points of interest.

COROLLARY 1. *Let D be an open subset of $\mathbf{C}^n$, and let α be a C^2 boundary point of D. If there is a "plurisubharmonic barrier" for α on D, then α is a limit* *of strictly pseudoconvex boundary points of D. (By a "plurisubharmonic barrier" we mean a function $u \in C^2_{\mathbf{R}}(D)$ such that* $\limsup_{z \in D, z \to \alpha} u(z) = 0$ *but for each neighborhood V of α in $\mathbf{C}^n$,* $\sup_{D \setminus V} u < 0$.)

PROOF. After a suitable change of coordinates the Theorem may be applied on a small neighborhood of α.

COROLLARY 2. *Suppose that D is an open subset of $\mathbf{C}^n$, that α is a C^2 boundary point of D, and that there is a function $f \in A(D)$ which peaks at α ($f(\alpha) = 1$ and $|f| < 1$ on $\bar{D} \setminus \{\alpha\}$). Then α is a limit of strictly pseudoconvex boundary points.*

PROOF. Apply Corollary 1 in a neighborhood of α with $u = \log|f|$.

COROLLARY 3. *Let D be a bounded domain in $\mathbf{C}^n$ with C^2 boundary. Then the Shilov boundary of $A(D)$ is a subset of the closure of the strictly pseudoconvex boundary points of $A(D)$.*

PROOF. The Shilov boundary of a uniform algebra on a compact metric space is the closure of the set of peak points. Thus Corollary 3 follows from Corollary 2.

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