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THE GROUP C^*-ALGEBRA OF THE DESITTER GROUP

ROBERT BOYER AND ROBERT MARTIN[1]

ABSTRACT. Let G denote the universal-covering of the DeSitter group and $C^*(G)$ the group C^*-algebra of G. In this paper we use the extension theory of C. Delaroche to describe the structure of $C^*(G)$.

Introduction. Let G denote the universal-covering of the DeSitter group and $C^*(G)$ the group C^*-algebra of G, i.e., the enveloping C^*-algebra of the involutive Banach algebra $L_1(G)$ (see [2]). The main goal of this paper is to give a complete description of the structure of $C^*(G)$. Briefly, the main result is that $C^*(G)$ is isomorphic to the restricted product of certain C^*-algebras whose structures have concrete descriptions given by the extension theory of C. Delaroche [1].

In §1 of this paper we summarize the classification of the irreducible unitary representations of G given by J. Dixmier [3] and the character formulas for these representations given by T. Hirai [6]. We refer to [3] or [9] for all information concerning the structure of G.

In §2 we investigate the behavior of the irreducible characters and then follow the method of J. M. G. Fell [4] to describe the topology on $\hat{G}$. An important step in this program is that of proving a Riemann-Lebesgue lemma for G. This we also do in §2.

In §3 we determine the structure of $C^*(G)$. Since there are an infinite number of points where $\hat{G}$ fails to be Hausdorff, the methods of [1] do not apply directly. However, we are able to express $C^*(G)$ as the restricted product of certain C^*-algebras each of which is describable via Theorem VI.3.8 of [1].

When $G = \mathrm{SL}(2, \mathbf{C})$, the structure of $C^*(G)$ was first described by Fell [5] and later by Delaroche [1]. For $G = \mathrm{SL}(2, \mathbf{R})$, the structure of $C^*(G)$ was determined by Miličić [7] by using methods similar to those of Fell in the $\mathrm{SL}(2, \mathbf{C})$ case. For the remaining Lorentz groups, one should be able to use the parameterization of $\hat{G}$ given by Thieleker [10], the character formulas given by Hirai [6], and the Delaroche extension theory to obtain results similar to those in this paper. This problem reduces to knowing the topological behavior at the "ends" of the complementary series representations,

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proving a Riemann-Lebesgue lemma for these groups, and then expressing $C^*(G)$ as a restricted product of C^*-algebras to which Theorem VI.3.8 of [1] is applicable.

1. **The representations and characters of** G. The DeSitter group is the group $G' = SO_e(4, 1)$, i.e., the identity component of the group of all automorphisms of $\mathbf{R}^5$ which preserve the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2$. G' is a connected semisimple real-rank one Lie group with trivial center. $G = \mathrm{Spin}(4, 1)$ is the simply-connected double covering of G'. Using the parameterization of $\hat{G}$ as a subset of $\mathbf{R}^2$ given by Dixmier [3], we have that the members of $\hat{G}$, other than the trivial representation I, fall into the following 4 categories:

A. *The continuous series* $\mathfrak{N}$. The collection of continuous series representations is $\mathfrak{N} = \{\gamma(n, s): s > -2 \text{ if } n = 0, s > 0 \text{ if } n = 1, 2, \ldots, \text{ and } s > 1/4 \text{ if } n = 1/2, 3/2, \ldots\}$ where $\gamma(n, s)$ is as in [3]. $\mathfrak{N}$ is the disjoint union of the irreducible principal series representations

$$\mathcal{P} = \{\gamma(n, s): s \geqslant 1/4 \text{ if } n = 0, 1, 2, \ldots \text{ and } s > 1/4 \text{ if } n = 1/2, \ldots\}$$

and the complementary series representations

$$\mathcal{C} = \{\gamma(n, s): -2 < s < 1/4 \text{ if } n = 0; 0 < s < 1/4 \text{ if } n = 1, 2, \ldots\}.$$

B. *The reducible principal series* $\mathfrak{R}$. These are the irreducible representations arising as summands of the reducible principal series representations. So

$$\mathfrak{R} = \{\pi^{\pm}(n, 1/2): n = 1/2, 3/2, \ldots\}.$$

C. *The discrete series* $\mathfrak{D}$. This is the collection $\mathfrak{D} = \{\pi^{\pm}(n, q): n = 1, 3/2, \ldots; q = n, n - 1, \ldots, 3/2 \text{ or } 1\}$.

D. *The end point representations* $\mathfrak{S}$. This, in Dixmier's notation, is the collection $\mathfrak{S} = \{\pi(n, 0): n = 1, 2, \ldots\}$.

We shall identify $\hat{G}$ with the following subset of $\mathbf{R}^2$: to each representation $\gamma(n, s)$ we associate the point (n, s); to the pair $\pi^{\pm}(n, q) \in \mathfrak{R} \cup \mathfrak{D}$ we associate a double point at $(n, -q)$; to $\pi(n, 0) \in \mathfrak{S}$ we associate the point $(n, 0)$ [note that these points occur as end points of the various intervals comprising the complementary series for $n = 1, 2, \ldots$]; and to I we associate the end point of the class one complementary series $(n = 0)$, $(0, -2)$.

Using the realization of G as a certain collection of two by two matrices over the quaternions given in [9], we let

$$m_u = \begin{pmatrix} e^{iu/2} & 0 \\ 0 & e^{iu/2} \end{pmatrix}, \quad m_v = \begin{pmatrix} e^{iv/2} & 0 \\ 0 & e^{-iv/2} \end{pmatrix}, \quad a_t = \begin{pmatrix} \mathrm{ch}\, t/2 & \mathrm{sh}\, t/2 \\ \mathrm{sh}\, t/2 & \mathrm{ch}\, t/2 \end{pmatrix},$$

$$A = \{a_t: t \in \mathbf{R}\}, \quad B = \{m_u: u \in \mathbf{R}\}, \quad \text{and} \quad T = \{m_u m_v: u, v \in \mathbf{R}\}.$$

Then $A_1 = BA$ and $A_2 = T$ are the nonconjugate Cartan subgroups of G. The character of each $\pi \in \hat{G}$ is given by integration against a locally summable function on G. We shall take the liberty of denoting both by the symbol Θ_π–so for $f \in C_c^\infty(G)$, $\Theta_\pi(f) = \int_G f(g)\Theta_\pi(g)\,dg$. Setting $G_i = \bigcup_{g\in G} gA_i g^{-1}$ for $i = 1, 2$, we recall that $G_1 \cup G_2$ is almost all of G (i.e., G up to a set of Haar measure zero) and that $\Theta_\pi(xh_ix^{-1}) = \Theta_\pi(h_i)$ for $h_i \in A_i$. We now recall the formulas for the irreducible characters of G given by Hirai [6]. For $h \in A_1$ we let $\Delta_1(h) = 2|\text{sh}\, t/2|(\sin u/2)(\text{ch}\, t - \cos u)$ and for $h \in A_2$ we let $\Delta_2(h) = 2(\sin v/2)(\sin u/2)(\cos v - \cos u)$.

A. For $\gamma(n, s) \in \mathcal{P}$ and $s = 1/4 + m^2$,

$$\Theta(n,s)(h) = \begin{cases} \dfrac{\cos(mt)\sin(n+1/2)u}{\Delta_1(h)} & \text{if } h \in A_1, \\ 0 & \text{if } h \in A_2. \end{cases}$$

B. For $\gamma(n, s) \in \mathcal{C}$ and $s = 1/4 - m^2$,

$$\Theta(n,s)(h) = \begin{cases} \dfrac{\text{ch}(mt)\sin(n+1/2)u}{\Delta_1(h)} & \text{if } h \in A_1, \\ 0 & \text{if } h \in A_2. \end{cases}$$

C. Letting $\Theta^\pm(n, 1/2)$ denote the character for $\pi^\pm(n, 1/2) \in \mathcal{R}$, we have

$$\Theta^+(n,1/2)(h) + \Theta^-(n,1/2)(h) = \begin{cases} \dfrac{\sin(n+1/2)u}{\Delta_1(h)} & \text{if } h \in A_1, \\ 0 & \text{if } h \in A_2. \end{cases}$$

D. Letting $\Theta^\pm(n, q)$ denote the character for $\pi^\pm(n, q) \in \mathcal{D}$, we have

$$\Theta^+(n,q)(h) + \Theta^-(n,q)(h)$$

$$= \begin{cases} \dfrac{\exp(-(q-1/2)|t|)\sin(n+1/2)u - \exp(-(n+1/2)|t|)\sin(q-1/2)u}{\Delta_1(h)} & \text{if } h \in A_1, \\ \dfrac{-\sin(q-1/2)v\,\sin(n+1/2)u + \sin(n+1/2)v\,\sin(q-1/2)u}{\Delta_2(h)} & \text{if } h \in A_2. \end{cases}$$

E.

$$\Theta^E(n,0)(h) = \begin{cases} \dfrac{\exp(-(n+1/2)|t|)\sin(u/2) + \sin(n+1/2)u\,\text{sh}(|t|/2)}{\Delta_1(h)} & \text{if } h \in A_1, \\ \dfrac{\sin(n+1/2)u\,\sin(v/2) - \sin(u/2)\sin(n+1/2)v}{\Delta_2(h)} & \text{if } h \in A_2. \end{cases}$$

2. **The topology on** $\hat{G}$. With the identification of $\hat{G}$ as given in §1 we have that $\hat{G} = \mathfrak{N} \cup \mathfrak{R} \cup \mathfrak{D} \cup \mathfrak{E} \cup \{I\} \subseteq \mathbf{R}^2$. Roughly speaking, to describe the topology of $\hat{G}$, we must determine the topology on each of the pieces $\mathfrak{N}$, $\mathfrak{R}$, $\mathfrak{D}$, $\mathfrak{E}$, and $\{I\}$ and then investigate how these pieces fit together topologically. We do this by investigating the various limits of the characters in conjunction with a key theorem due to Fell [4, p. 391].

LEMMA 1. (1) *Let* $J_0 = (-2, \infty)$, $J_n = (0, \infty)$ *for* $n = 1, 2, \ldots,$ *and* $J_n = (1/4, \infty)$ *for* $n = 1/2, \ldots.$ *Then if* $\{s_\sigma\}$ *is a sequence in* J_n *for* $n = 0, 1/2, 1, \ldots$ *with* $s_\sigma \to s \in J_n$, $\Theta(n, s_\sigma) \to \Theta(n, s)$.

(2) *If* $n = 1/2, 3/2, \ldots$ *and* $\{s_\sigma\}$ *is a sequence in* $(1/4, \infty)$ *with* $s_\sigma \to 1/4$, *then* $\Theta(n, s_\sigma) \to \Theta^+(n, 1/2) + \Theta^-(n, 1/2)$.

(3) *If* $\{s_\sigma\}$ *is a sequence in* $(-2, \infty)$ *with* $s_\sigma \to -2$, *then* $\Theta(0, s_\sigma) \to 1 + \Theta^E(1, 0)$.

(4) *If* $n = 1, 2, \ldots$ *and* $\{s_\sigma\}$ *is a sequence in* $(0, \infty)$ *with* $s_\sigma \to 0$, *then* $\Theta(n, s_\sigma) \to \Theta^E(n, 0) + \Theta^+(n, 1) + \Theta^-(n, 1)$.

PROOFS. Direct application of the character formulas.

LEMMA 2 (RIEMANN-LEBESGUE LEMMA FOR G). *If* $\{\pi_\gamma\}$ *is a sequence in* $\hat{G}$ *whose underlying parameters tend to* ∞ *in* $\mathbf{R}^2$ *and* $f \in C_c^\infty(G)$, *then* $\Theta_{\pi_\gamma}(f) \to 0$.

PROOF. First we note that it suffices to consider sequences in either $\mathfrak{P} \cup \mathfrak{R}$, $\mathfrak{D}$, $\mathcal{C}$, or $\mathfrak{E}$. For a sequence in $\mathfrak{P} \cup \mathfrak{R} \cup \mathfrak{D}$, this result has been proven by R. Lipsman (see [11]), and so we need only consider the cases $\mathcal{C}$ or $\mathfrak{E}$. For a sequence $\{\pi_\gamma\}$ in $\mathcal{C}$ to converge to infinity, we must have that the underlying parameters (n_γ, s_γ) satisfy $n_\gamma \to \infty$ and $|s_\gamma| < 2$. We now use the fact that (see [11, Chapter 8] and [8]) it is possible to normalize the invariant measure $d_1\bar{g}$ on G/A_1 such that for $f \in C_c^\infty(G)$,

$$\int_{G_1} f(g)\,dg = \int_{A_1}\int_{G/A_1} f(ghg^{-1})\Delta_1^2(h)\,d_1\bar{g}\,dh$$

where dg is Haar measure on G and dh is Haar measure on A_1 $(= S^1 \times \mathbf{R}^+)$-(our $\Delta_1(h)$ differs by a constant from that of [8] which we have chosen to place in the measure $d_1\bar{g}$). If we write $F_f^1(h) = \Delta_1(h)\int_{G/A_1} f(ghg^{-1})\,d_1\bar{g}$ for $f \in C_c^\infty(G)$, then F_f^1 has compact support on A_1 and

$$\Theta(n_\gamma, s_\gamma)(f) = \int_{A_1}\int_{G/A_1} f(ghg^{-1})\Theta(n_\gamma, s_\gamma)(ghg^{-1})\Delta_1^2(h)\,d_1\bar{g}\,dh$$
$$= \int_{A_1} F_f^1(h)\operatorname{ch}(m_\gamma t)\sin(n_\gamma + 1/2)u\,dh.$$

Since the m_γ's are bounded, the desired result now follows from a simple uniformity argument together with the Riemann-Lebesgue lemma for the locally compact abelian group A_1.

Now let $\{\pi(n_\gamma, 0)\}$ be a sequence in $\mathfrak{E}$ with $n_\gamma \to \infty$. Normalize the invariant measure $d_2\bar{g}$ on G/A_2 such that for $f \in C_c^\infty(G)$,

$$\int_{G_2} f(g)\,dg = \int_{A_2}\int_{G/A_2} f(ghg^{-1})\Delta_2^2(h)\,d_2\bar{g}\,dh$$

where dh is Haar measure on A_2 $(= S^1 \times S^1)$ (again our Δ_2 is a constant times that appearing in [8]). For $f \in C_c^\infty(G)$ we have that

$$F_f^i(h_i) = \Delta_i(h_i)\int_{G/A_i} f(gh_ig^{-1})\,d_i\bar{g}$$

has compact support on A_i. Then

$$\Theta^E(n_\gamma, 0)(f) = \sum_{i=1}^{2}\int_{A_i}\int_{G/A_i} f(gh_ig^{-1})\Theta^E(n_\gamma, 0)\Delta_i^2(h_i)\,d_i\bar{g}\,dh_i$$

$$= \sum_{1}^{2}\int_{A_i} F_f^i(h_i)\Theta^E(n_\gamma, 0)(h_i)\Delta_i(h_i)\,dh_i$$

$$= \int_{A_1} F_f^1(h_1)\big(\exp(-(n_\gamma + 1/2)|t|)\sin(u/2) + \sin(n_\gamma + 1/2)u\ \mathrm{sh}|t/2|\big)\,dh_1$$

$$+ \int_{A_2} F_f^2(h_2)\big(\sin(n_\gamma + 1/2)u\sin(v/2) - \sin(u/2)\sin(n_\gamma + 1/2)v\big)\,dh_2.$$

Using three Riemann-Lebesgue lemma arguments and one Lebesgue dominated convergence theorem argument, we may conclude that $\Theta^E(n_\gamma, 0)(f) \to 0$ as $n_\gamma \to \infty$.

The above lemmas, [4, p. 391], and the fact that points are closed (G is CCR) completely determine the topology on $\hat{G}$. Part (1) of Lemma 1 implies that the topology on $\mathfrak{N}$ is identical to that it inherits as a subset of $\mathbf{R}^2$ (and hence is T_2). Lemma 2 implies that all limits of sequences in $\hat{G}$ occur in the finite plane. This, together with the fact that points are closed, implies that $\mathfrak{R} \cup \mathfrak{D} \cup \mathfrak{S} \cup \{I\}$ is closed in $\hat{G}$. Parts (2), (3), and (4) of Lemma 1 tell us how the pieces fit together. They do so in the following way: (1) the closure of any subset of $\mathfrak{N}$ that would ordinarily contain the point $(0, -2)$ must contain both $(0, -2)$ and $(1, 0)$; (2) the closure of any subset of $\mathfrak{N}$ that would ordinarily contain $(n, 1/4)$ for $n = 1/2, \ldots$ must contain the pair of points at $(n, -1/2)$; and (3) the closure of any subset of $\mathfrak{N}$ that would ordinarily contain $(n, 0)$ for $n = 1, 2, \ldots$ must contain the pair of points at $(n, -1)$ in addition to the point $(n, 0)$. We also note:

(1) on each of the pieces $\mathfrak{N}, \mathfrak{R}, \mathfrak{D}, \mathfrak{S}, \{I\}$ the topology coincides with the natural topology of the underlying parameter space–so on each separate piece the topology is T_2;

(2) the pieces fit together in a non-T_2 manner (recall that $\hat{G}$ is T_1);

(3) $\mathfrak{P} \cup \mathfrak{R}$ is closed in $\hat{G}$ while $\mathfrak{P}$ and $\mathfrak{N}$ are not;

(4) $\hat{G}_r = \mathfrak{P} \cup \mathfrak{R} \cup \mathfrak{D}$ is closed in $\hat{G}$ but is not T_2;

(5) the collection $\{\pi^{\pm}(n, q): q > 3/2\}$ is both open and closed in $\hat{G}$;

(6) any subset of $\mathfrak{R} \cup \mathfrak{D} \cup \mathfrak{S} \cup \{I\}$ is closed in $\hat{G}$;

(7) $\mathfrak{D}$ is not open in $\hat{G}$ but is open (and closed) in $\hat{G}_r$; and

(8) the only point in $\hat{G}$ which fails to be separated from the trivial representation is $\pi(1, 0)$.

In summary we have

THEOREM 1. *Let* $P_0 = (0, -2)$, $P_n = (n, 0)$ *for* $n = 1, 2, \dots$, $P_n = (n, 1/4)$ *for* $n = 1/2, \dots$, $Q = \{P_0, P_{1/2}, P_1, \dots\}$, *and* $S \subseteq \hat{G}$. *Denote by* $\bar{S}$ *the (hull-kernel) closure of* S *in* $\hat{G}$ *and by* S_0 *the closure of* S *as a subset of* $\mathbf{R}^2$. *Then*

(i) *if* $P \in \hat{G} - (\{P_1\} \cup \{\pi^{\pm}(n, 1/2): n = 1/2, 1, \dots\})$, $P \in \bar{S}$ *iff* $P \in S_0$;

(ii) *if* $P = \pi^+(n, 1/2)$, $P \in \bar{S}$ *iff* $P \in S_0$ *or* $P_n \in S_0$;

(iii) *if* $P = P_1$, $P \in \bar{S}$ *iff* $P \in S_0$ *or* $P_0 \in S_0$.

3. **The structure of** $C^*(G)$. We now use the concept of a restricted product together with the extension theory of C^*-algebras given in [1] to determine the isomorphism type of the group C^*-algebra D of Spin(4, 1). We shall denote the representations in $\mathfrak{N} \cup \mathfrak{S} \cup \{I\}$ by their underlying parameters. We let H be a separable infinite-dimensional Hilbert space and $\mathcal{K}(H)$ the compact operators on H. If $S \subseteq \mathbf{R}^2$, let $(S)_1$ be the compactification obtained by forming the one-point compactification of the closure of S in $\mathbf{R}^2$ with point at infinity x_∞.

The dual space of D naturally decomposes into countably many components of three distinct types:

(1) $Z_0 = \{(0, s): s \geq -2\} \cup \{(1, s): s \geq 0\} \cup \{\pi^{\pm}(1, 1)\}$;

(2) $Z_i = \{(i, s): s > 1/4\} \cup \{\pi^{\pm}(i, 1/2)\} \cup \{\pi^{\pm}(i, q): q = i, i - 1, \dots, 3/2\}$, where $i \in M_1 = \{1/2, 3/2, \dots\}$;

(3) $Z_j = \{(j, s): s \geq 0\} \cup \{\pi^{\pm}(j, q): q = j, j-1, \dots, 1\}$, where $j \in M_2 = \{2, 3, \dots\}$.

Let I_k be the closed two-sided ideal of D with $\hat{I}_k = Z_k$, where $k \in M = M_1 \cup M_2$. First we describe the C^*-structure of these ideals, then explain how D is determined. We shall only indicate what to define in order to apply the theorems of [1, Chapitre VI].

The description of I_0: Let $X = Z_0 - [\{P_0\} \cup \{P_1\} \cup \{\pi^{\pm}(1, 1)\}]$, and $Y = Z_0 - X$. The ideal A of D with $\hat{A} = X$ is isomorphic to $C^0(X, \mathcal{K}(H))$, the norm-continuous functions of X to $\mathcal{K}(H)$ vanishing at infinity, by [2, p. 219] since A has continuous trace by Lemma 1. If $C = \bigoplus_{i=1}^4 \mathcal{K}(H) \oplus \mathbf{C}$, then I_0 is an extension of A by C. Let f: $(X)_1 - X \to \mathcal{F}(Y)$ by $f(x_\infty) = \varnothing$, $f(P_0) = \{P_0\} \cup \{P_1\}$, $f(P_1) = \{P_1\} \cup \{\pi^{\pm}(1, 1)\}$. Then I_0 is the extension of X by Y associated with f. We apply the generalization [1, VI. 3.9] of Theorem VI. 3.8 of [1] where $\Omega^1 = \{P_0\}$, $\Omega^2 = \{P_1\}$, $n = 2$, $q_1 = 2$, $q_2 = 3$, $s_1 = s_2 = 1$. Moreover $k_1^i(j) = 1$ for $1 \leq j \leq q_i$, $i = 1, 2$, by parts (2) and (3) of Lemma 1. Identify H with $\bigoplus_{i=1}^4 H \oplus \mathbf{C}$. Then I_0 is isomorphic to the C^*-algebra of pairs $(m, c_1 \oplus c_2 \oplus c_3 \oplus c_4 \oplus \eta) \in C^b(X, \mathcal{K}(H)) \times C$ (where $C^b(X, \mathcal{K}(H))$ denotes the bounded norm-continuous functions of X to $\mathcal{K}(H)$) such that $\lim_{t \to P_0} m(t) = c_1 \oplus 0 \oplus 0 \oplus 0 \oplus \eta$, $\lim_{t \to P_1} m(t) = 0 \oplus c_1 \oplus c_2 \oplus c_3 \oplus 0$, and $\lim_{t \to \infty} m(t) = 0$.

The description of I_k, $k \in M$: Let $X_i = Z_i - \{\pi^{\pm}(i, 1/2)\}$, $Y_i = \{\pi^{\pm}(i, 1/2)\}$, $i \in M_1$; $X_j = Z_j - [\{\pi^{\pm}(j, 1)\} \cup \{P_j\}]$, $Y_j = \{\pi^{\pm}(j, 1)\} \cup$

$\{P_j\}, j \in M_2$. The ideal $A_k, k \in M$, of D with $\hat{A}_k = X_k$ is isomorphic to $C^0(X_k, \mathcal{K}(H))$ since A_k has continuous trace by Lemma 1. Let $N(k) = 2$ if $k \in M_1$ and $N(k) = 3$ if $k \in M_2$. If $C = \oplus_{i=1}^{N(k)}\mathcal{K}(H)$, then I_k is an extension of A_k by C. Let f: $(X_k)_1 - X_k \to \mathcal{F}(Y_k)$ by $f(x_\infty) = \varnothing, f(P_k) = Y_k, k \in M$. Then $\hat{I}_k = Z_k$ is the extension of X_k by Y_k associated with f. We apply [1, VI. 3.8] where $\Omega^1 = \{P_k\}, n = 1, q_1 = N(k), s_1 = 1$. Moreover $k_1^1(r) = 1$ for $1 \leqslant r \leqslant N(k)$ by parts (2) and (4) of Lemma 1. Identify H with $\oplus_{i=1}^{N(k)}H$. Then I_k is isomorphic to the C^*-algebra of pairs $(m, c) \in C^b(X_k, \mathcal{K}(H)) \times C$ such that $\lim_{t\to P_k} m(t) = c_1 \oplus c_2$ if $k \in M_1$, $\lim_{t\to P_k} m(t) = c_1 \oplus c_2 \oplus c_3$ if $k \in M_2$, and $\lim_{t\to\infty} m(t) = 0$.

We next show that D is the restricted product [2, 1.9.4] of the ideals $I_k, k \in M \cup \{0\}$.

LEMMA 3. *Let $\mathfrak{a}$ be a C^*-algebra without identity. If $\hat{\mathfrak{a}} = \bigcup_{n=1}^{\infty} X_n$, where the X_n are disjoint nonempty open subsets of $\hat{\mathfrak{a}}$, then $\mathfrak{a}$ is isomorphic to the restricted product B of the ideals I_n where $\hat{I}_n = X_n$.*

PROOF. Consider the ideal $\mathfrak{c}$ of $\mathfrak{a}$, where $\mathfrak{c} = \overline{\bigcup_{k=1}^{\infty} \oplus_{n=1}^{k} I_n}$. It is easy to see that for any $\pi \in \hat{\mathfrak{a}}$, $\pi(\mathfrak{c}) \neq 0$. Thus, $\mathfrak{c} = \mathfrak{a}$ by [2, 3.2.2]. We now map $\mathfrak{c}$ onto the restricted product B in the obvious way.

The following theorem now follows immediately.

THEOREM 2. *If G is the universal covering of the DeSitter group, then $C^*(G)$ is isomorphic to the restricted product of the ideals I_k, $k \in M \cup \{0\}$.*

Theorem 2 determines the isomorphism type of $C^*(G)$. It is, of course, possible to obtain alternate descriptions for the structure of $C^*(G)$; for example, it follows from [2, 10.10.2] that $C^*(G)$ is isomorphic to $I_0 \oplus I_{1/2} \oplus C^0(\mathbf{Z}^+, J)$ where $J = I_{3/2} \oplus I_2$. One may also obtain descriptions similar to those in [5] and [7].

One can easily show that if G' is the DeSitter group, then $\hat{G}' = Z_0 \cup \bigcup_{j\in M_2} Z_j$ with the relative topology it inherits as a subset of $\hat{G}$. Thus we have

THEOREM 2′. *Let G' be the DeSitter group $SO_e(4, 1)$. Then $C^*(G')$ is isomorphic to the restricted product of the ideals I_k, $k \in M_2 \cup \{0\}$.*

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Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19174 (Current address of Robert Boyer)

Current address (Robert Martin): Department of Mathematics, Middlebury College, Middlebury, Vermont 50753