ISOTONE FUNCTIONS ON PARTIALLY ORDERED LINEAR ALGEBRAS WITH A MULTIPLICATIVE DIAGONAL MAP

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ABSTRACT. The diagonal of the product of two triangular matrices is the product of the diagonals of each matrix. This idea is used to characterize Dedekind σ-complete lattice ordered linear algebras which admit isotone functions with familiar functional and order properties as possessed by the real-valued logarithm or root function.

1. Motivated by the work of Kadison and Singer [5] on triangular operator algebras, the authors [3] characterized abstract partially ordered linear algebras (polas) which have the order properties similar to an algebra of real upper triangular matrices. The main idea of the characterization we used in [3] is that in an algebra of upper triangular matrices the diagonal of the product of two matrices is equal to the product of the diagonals. We use the same idea in this paper to characterize the polas which admit an isotone function with familiar functional and order properties as possessed by the real-valued logarithm function or root function on the real line.

A dsc-pola, denoted by A, is a real linear associative algebra which satisifes the following two conditions: (1) It is partially ordered so that it is a directed partially ordered linear space and $0 \leqslant xy$ whenever $0 \leqslant x, y \in A$. (2) It is Dedekind σ-complete, i.e., if $x_n \in A$, $x_1 \geqslant x_2 \geqslant \cdots \geqslant 0$, then $\inf\{x_n\}$ exists. Order convergence can be defined as usual. In this paper, we assume A has a multiplicative identity $1 > 0$. Let $I = \{y \in A: 1 \leqslant y$ and $y^{-1} > 0\}$. Define $A_1 = I - I$. It is shown in [2] that A_1 is an order-closed and order-convex commutative subalgebra of A which behaves much like an algebra of real-valued functions. A_1 is called the *diagonal* or *functional* part of A. For detailed discussions and examples of A_1, see [2]; there we used the term polac instead of dsc-pola. A general discussion of lattices and lattices of operators may be found in [1] and [7].

Next we define a diagonal projection (*DP*) map Δ: $A \to A_1$ to be a linear map such that $\Delta(1) = 1$ and if $x \geqslant 0$, then $x \geqslant \Delta(x) \geqslant 0$. Since Δ is linear, it follows trivially that Δ is isotone, i.e., $\Delta(y) \geqslant \Delta(z)$ whenever $y \geqslant z, y, z \in A$. This map was studied in [3]. Example 2.4 in [3] shows that in some dsc-pola there may not exist a *DP* map; however, if it exists, it is unique [3,

Theorem 3.3]. But, when A is a lattice, there always exists a DP map [2, p. 671]. For convenience, we state the construction briefly. For any positive $x \in A$, define $\Delta(x) = \sup\{x \wedge n1\}$. Since $0 \leqslant x \wedge n1 \leqslant n1$, as well as $x \wedge n1 \leqslant x$, and since A_1 is order-convex and order-closed, we have $x \wedge n1 \in A_1$ and $\Delta(x) \in A_1$. For any $y \in A$, define $\Delta(y) = \Delta(y^+) - \Delta(y^-)$. ($y^+ = y \vee 0$, $y^- = (-y) \vee 0$.) Then $\Delta(y) \in A_1$. This map Δ is a diagonal projection map. (In [2, p. 671], notation $d(\cdot)$, rather than $\Delta(\cdot)$, was used.) Also, the reader should note that the concept of a DP map is related to the so-called "diagonal process" defined by Kadison and Singer; see [4, p. 387].

Examples of a DP map will be given in §2. We now state some important basic properties of a DP map.

THEOREM 1.1. *Suppose a DP map Δ exists on A. Then*
(i) *if $a \in A_1$, then $\Delta(a) = a$;*
(ii) *if $a \in A_1$, then $\Delta(ax) = \Delta(xa) = a\Delta(x)$ for $x \in A$; and*
(iii) *if $x, y \in A$, $0 \leqslant x$, $0 \leqslant y$, then $0 \leqslant \Delta(x)\Delta(y) \leqslant \Delta(xy)$.*

PROOF. See [3, Theorem 3.2]. □

Example 2.1 in the following section clearly indicates that (iii) is, in general, a strict inequality. In case that $\Delta(xy) = \Delta(x)\Delta(y)$ for all $x, y \in A$, we say Δ is multiplicative. In §3, we shall show that the admittance of some special kind of isotone functions on A implies Δ is multiplicative.

2. Some elementary examples will be given to illustrate the basic concepts involved in this paper. For more examples of a diagonal projection map, we refer to [3, §2].

EXAMPLE 2.1. Let A be the algebra of all the real n-by-n matrices with entry by entry ordering. Then A is a dsc-pola (also a lattice). A_1 is nothing but all the diagonal matrices of A. For $y = [\alpha_{ij}] \in A$, define $\Delta(y)$ to be the diagonal part of y. It is trivial to verify that Δ is the DP map, but Δ is not multiplicative.

EXAMPLE 2.2. Let A be the algebra of all the real n-by-n upper triangular matrices with entry by entry ordering. Define Δ as in Example 2.1 above. The reader should note that Δ is a multiplicative DP map in this example.

On certain class of dsc-polas, there always exists some isotone function having the familiar functional and order properties as possessed by the real-valued logarithm or root function on the line; see Remark 3.8. We describe those functions precisely as follows. Let $H = \{x \in A: 1 \leqslant x\}$. An isotone function $f: H \to A$ is said to have the logarithm property if it satisfies the following three conditions:
(1) $f(xy) = f(x) + f(y)$ for $x, y \in H$;
(2) $f(x) \leqslant x - 1$; and
(3) there exists $a \in H$ such that $f(a) \geqslant 1$.

Similarly, an isotone function $g: H \to A$ is said to have the root property if it meets the following three conditions:
(1) $g(xy) = g(x)g(y)$;

(2) $g(x) - 1 \leqslant \lambda(x - 1)$ for a fixed λ, $1 > \lambda > 0$;

(3) there exists $a \in H$ such that $1 \leqslant g(a) - 1$.

Note that (2) in the definition of f (resp. g) is a standard inequality for the usual logarithm (resp. root) function. Without (3) we could get a trivial f (resp. g) by defining $f(x) = 0$ (resp. $g(x) = 0$ or $g(x) = 1$) for all $x \in H$.

EXAMPLE 2.3. Let A be as in Example 2.2 above with $n = 2$. For $1 \leqslant x = \left(\begin{smallmatrix} \alpha & \nu \\ 0 & \beta \end{smallmatrix} \right)$, define

$$f(x) = \begin{pmatrix} \log_3 \alpha & 0 \\ 0 & \log_4 \beta \end{pmatrix}.$$

Then we can check easily that f is a function with the logarithm property.

EXAMPLE 2.4. Let A be as in Example 2.3. For $1 \leqslant x = \left(\begin{smallmatrix} \alpha & \nu \\ 0 & \beta \end{smallmatrix} \right)$, define

$$g(x) = \begin{pmatrix} \alpha^{1/2} & 0 \\ 0 & \beta^{1/3} \end{pmatrix}.$$

Choose $\lambda = 1/2$. Again, it is easy to check that g is a function with the root property.

3. In [3], various algebraic (order) conditions on A were given which imply that a diagonal projection map is multiplicative. In this paper, we first prove the following theorem.

THEOREM I. *Suppose a lattice dsc-pola A admits a function f with the logarithm property. Then the diagonal projection map Δ is multiplicative.*

The theorem will be proved with the aid of a series of lemmas. Observe $f(1) = f(1) + f(1)$ gives $f(1) = 0$.

LEMMA 3.1. *If $x \in H$, then $f(x) \in A_1$.*

PROOF. By isotonicity and (1), (2) in the definition of f, we have

$$0 = f(1) \leqslant f(x) \leqslant f(n1 + x) = f(n1) + f(1 + n^{-1}x)$$

$$\leqslant (n - 1)1 + (1 + n^{-1}x - 1) = (n - 1)1 + n^{-1}x$$

for all positive integers n.

Since A is lattice, we may write $f(x) = x_n + w_n$, where $0 \leqslant x_n \leqslant (n - 1)1$ and $0 \leqslant w_n \leqslant n^{-1}x$ for all n. From the fact that A_1 is order-convex and order-closed, we have $x_n \in A_1$ and $f(x) = \lim_{n \to \infty} x_n \in A_1$. $\square$

LEMMA 3.2. $f(x) = f(\Delta(x))$ for $x \in H$.

PROOF. Since $1 \leqslant x$ and Δ is a DP map, we have $1 \leqslant \Delta(x) \leqslant x$. This, by the isotone property of f, implies $f(\Delta(x)) \leqslant f(x)$. Next, observe $\Delta(x) \in A_1 \cap H$. By the definition of A_1 and Lemma I.6.6(ii) of [2], we know $\Delta(x)^{-1} > 0$. Thus if $y = \Delta(x)^{-1}x$, then $1 \leqslant y$ and $\Delta(y) = 1$ (Theorem 1.1(ii)). From $x = \Delta(x)y$, it follows that $f(x) = f(\Delta(x)) + f(y) \leqslant f(\Delta(x)) + y - 1$. Now let us apply Δ to both sides of this inequality. Since both $f(x)$ and $f(\Delta(x))$ belong to A_1, using Thoerem 1.1(i), we obtain $f(x) \leqslant f(\Delta(x))$. Therefore, $f(x) = f(\Delta(x))$. $\square$

LEMMA 3.3. *If* $1 \leqslant z \in A_1$ *and* $f(z) = 0$, *then* $z = 1$.

PROOF. Note that $f(z^n) = nf(z) = 0$; and $1 \leqslant z \in A_1$ implies $0 < z^{-n}$ for all positive integers n. Now by the definition of f, there exists $a \in H$ such that $f(a) > 1$. Together, those facts show that

$$1 < f(a + z^n) = f(z^n) + f(1 + z^{-n}a) \leqslant 1 + z^{-n}a - 1 = z^{-n}a.$$

Therefore, $1 \leqslant z^n \leqslant a$ for all n. This means $z = 1$.   □

PROOF OF THEOREM I. For $x, y \in H$, using Lemma 3.2 twice, we have

$$f(\Delta(x)\Delta(y)) = f(\Delta(x)) + f(\Delta(y)) = f(x) + f(y)$$
$$= f(xy) = f(\Delta(xy)).$$

Let $c = \Delta(x)\Delta(y), d = \Delta(xy)$. Then $1 \leqslant c \leqslant d \in A_1$ by Theorem 1.1(iii). Thus, $1 \leqslant c^{-1}d$. From the relation $f(d) = f(c) + f(c^{-1}d)$ and the above result, it follows that $f(c^{-1}d) = 0$. By Lemma 3.3, $c^{-1}d = 1$ or $c = d$, i.e., $\Delta(xy) = \Delta(x)\Delta(y)$ for $x, y \in H$. Since A is directed and Δ is linear, it is straightforward to verify that Δ is multiplicative.   □

THEOREM II. *Suppose a lattice dsc-pola A admits a function g with the root property. Then the diagonal projection map Δ is multiplicative.*

Since the proof of this theorem is similar to the proof of the previous theorem, we only sketch the procedure.

LEMMA 3.4. $g(n1) \in A_1$, *and* $\lim_{k \to \infty}[g(n_k 1)/n_k] = 0$ *for a subsequence* $\{n_k\}$.

PROOF. We first show $g(1) = 1$. From the definition of g, it follows that $g(1) \leqslant 1$ and $g(a) = g(a)g(1) \geqslant 2(1)$. By [2, Theorem I.6.3], $g(1)$ has an inverse. This and the fact that $g(1) = g(1)^2$ yield $g(1) = 1$. By isotonicity, we have $1 = g(1) \leqslant g(n1) \leqslant 1 + \lambda(n-1)1$. Consequently, $g(n1) \in A_1$. Let k be an arbitrary positive integer. Then

$$g(n1) = \left(g(\sqrt[k]{n}\, 1)\right)^k \leqslant \left(1 + \lambda(\sqrt[k]{n} - 1)1\right)^k.$$

Thus

$$n^{-1}g(n1) \leqslant \left((1-\lambda)n^{-1/k}1 + \lambda 1\right)^k.$$

In particular, when $n = n_k = k^k$, we have

$$0 \leqslant (n_k)^{-1}g(n_k 1) \leqslant \left((1-\lambda)k^{-1}1 + \lambda 1\right)^k.$$

Finally, $1 > \lambda > 0$ implies $\lim_{k \to \infty}(n_k)^{-1}g(n_k 1) = 0$.   □

LEMMA 3.5. $g(x) \in A_1$ *for* $x \in H$.

PROOF. Since

$$1 \leqslant g(x) \leqslant g(x + n1) = g(n1)g(1 + n^{-1}x)$$
$$\leqslant g(n1)\left(1 + \lambda(1 + n^{-1}x - 1)\right) = g(n1) + (\lambda/n)g(n1)x,$$

we have from Lemma 3.4 that there exists a subsequence $\{n_k\}$ such that $1 \leqslant g(x) \leqslant g(n_k 1) + (\lambda/n_k) g(n_k 1)x$ and $\lim_{k\to\infty}(n_k)^{-1}g(n_k 1) = 0$. Since A is a lattice, we can proceed as in Lemma 3.1 to conclude $g(x) \in A_1$. □

LEMMA 3.6. $g(x) = g(\Delta(x))$ *for* $x \in H$.

PROOF. Since $1 \leqslant \Delta(x) \leqslant x$, we have $g(\Delta(x)) \leqslant g(x)$. Let $y = \Delta(x)^{-1}x \geqslant 1$. Then it follows as in the proof of Lemma 3.2 that $g(x) = g(\Delta(x))$. □

LEMMA 3.7. *If* $1 \leqslant z \in A_1$ *and* $g(z) = 1$, *then* $z = 1$.

PROOF. Note that $g(z^n) = g(z)^n = 1$. Then by the definition of g, we have
$$1 \leqslant g(a + z^n) - 1 = g(z^n)g(1 + z^{-n}a) - 1 \leqslant \lambda z^{-n}a,$$
thus $1 \leqslant z^n \leqslant \lambda a$ for all n. This implies $z = 1$. □

The proof of Theorem II can be completed by noting that if we let $c = \Delta(x)\Delta(y)$, $d = \Delta(xy)$ for $x, y \in H$, then it follows as in the proof of Theorem I that $g(c) = g(d)$. But, $1 \leqslant g(c) \in A_1$ and $g(d) = g(c)g(c^{-1}d)$ implies $g(c^{-1}d) = 1$. Hence, by Lemma 3.7, $c = d$. Thus, Δ is multiplicative. □

We now close the article by making the following comment.

REMARK 3.8. Any dsc-pola A with a multiplicative DP map Δ always admits a function with the logarithm (root) property which we explain as follows. In [6] it was proved that A_1 is algebraically and order isomorphic to an algebra S of continuous, almost-finite, extended-real-valued functions defined on a compact Hausdorff space. Hence, we may identify A_1 with S. It is clear then for $1 \leqslant z \in A_1$, $\ln(z)$ and $\sqrt{z}$ also belong to A_1. For $1 \leqslant x \in A$, we define $f(x) = \ln(\Delta(x))$ and $g(x) = \sqrt{\Delta(x)}$. It can be verified easily that f and g are functions with the logarithm property and the root property, respectively.

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