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REFLEXIVITY OF TOPOLOGICAL GROUPS

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ABSTRACT. It is shown that under mild conditions the path-component of the identity in the dual group $G^\wedge$ of an Abelian topological group G is precisely the union of the one-parameter subgroups of $G^\wedge$. This yields several corollaries, including a necessary condition for certain groups to be reflexive (to satisfy the Pontrjagin duality theorem), and a negative answer to a question of N. Noble.

The main result of this paper is a theorem on the structure of the dual group $G^\wedge$ of an Abelian topological group G; we show (under mild restrictions) that the path-component of the identity in $G^\wedge$ is precisely the union of the one-parameter subgroups of $G^\wedge$.

Several authors have studied the class of Abelian topological groups which are reflexive (that is, which satisfy the Pontrjagin duality theorem): it is known, for example, that the class contains many nonlocally compact groups ([1], [7], [8], [14], [16], [17]). It appears, however, that a few general necessary conditions for reflexivity are known. We derive such a condition here as a consequence of the main theorem, and then explore some of its applications. In particular, we show that many free Abelian topological groups fail to be reflexive, and we settle a question of N. Noble [14] in the negative.

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We shall use additive notation for all our groups except the circle group T, which we regard as the (compact) multiplicative group of complex numbers of modulus one. As usual, the dual group $G^\wedge$ of an Abelian topological group G is the group of continuous characters of G, with the compact-open topology. All topological groups and spaces considered here will be assumed Hausdorff.

THEOREM. *If the Abelian topological group G is a k-space, then the path-component of the identity in $G^\wedge$ is the union of all the one-parameter subgroups of $G^\wedge$.*

PROOF. The union of the one-parameter subgroups of $G^\wedge$ is clearly

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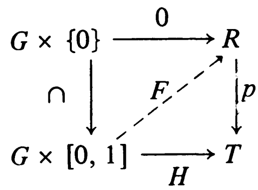
contained in the path-component P of the identity, and so only the reverse inclusion requires proof.

Fix a character $\gamma \in P$. We wish to construct a one-parameter subgroup of $G^{\wedge}$ in which γ lies, and we start the construction by lifting γ to a real character β: $G \to R$; this part of the argument is similar to that of (4.73) of [5]. Let f: $[0, 1] \to G^{\wedge}$ be a path joining 0 (the trivial character) and γ–thus $f(0) = 0$ and $f(1) = \gamma$. If we regard $G^{\wedge}$ as a subspace of the function space T^G with the compact-open topology, f may be regarded as a map from $[0, 1]$ into T^G, and, since G is a k-space, (XV, 3.1) of [2] shows that f induces a homotopy H: $G \times [0, 1] \to T$ between the trivial character 0 and γ. Then the homotopy lifting property (§2.2 of [15]) applied to the covering projection p: $R \to T$ defined by $p(x) = \exp(2\pi i x)$ shows that in the following commutative square a homotopy F can be found making the resulting triangles commute.



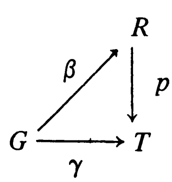
For any elements $g_1, g_2 \in G$ consider the map F': $[0, 1] \to R$ defined by

$$F'(x) = F(g_1, x) + F(g_2, x) - F(g_1 + g_2, x),$$

for $x \in [0, 1]$. It is clear that F' is continuous, that $F'(0) = 0$, and that F' is a lifting of the map H': $[0, 1] \to T$ given by

$$H'(x) = H(g_1, x)H(g_2, x)H(g_1 + g_2, x)^{-1}.$$

But since H is induced by f: $[0, 1] \to G^{\wedge}$, we see that $H'(x) = 1$ for all x, and F' is thus a lifting of the trivial path in T. The trivial path in R is also such a lifting, so since $F'(0) = 0$, the unique path lifting property of p: $R \to T$ [15, 2.2] implies $F'(x) = 0$ for all x. Hence, for $x \in [0, 1]$, $g_1, g_2 \in G$, we have $F(g_1 + g_2, x) = F(g_1, x) + F(g_2, x)$. In particular, $\beta = F|_{G \times \{1\}}$: $G \to R$ is a continuous homomorphism, making the following triangle commute.



Denote by $\mathrm{Hom}(G, R)$ the group of continuous homomorphisms of G into R given the compact-open topology and define ϕ: $R \to \mathrm{Hom}(G, R)$ by $\phi(x) = x\beta$, where $(x\beta)(g) = x(\beta g)$ for $g \in G$. It is easy to check that ϕ is a continuous homomorphism. Composition with the continuous homomorphism p_*: $\mathrm{Hom}(G, R) \to \mathrm{Hom}(G, T) = G^{\wedge}$ induced by p then gives us $\psi = p_*\phi$: $R \to G^{\wedge}$, and $\psi(R)$ is a one-parameter subgroup of $G^{\wedge}$. Finally, since $\psi(1) = p_*(\phi(1)) = p_*(\beta) = p\beta = \gamma$, we see that γ lies in $\psi(R)$ and the proof is complete.

Since R is a divisible group the following corollary is immediate.

COROLLARY 1. *With G as in the theorem, the path-component of the identity in $G^\wedge$ is divisible.*

The corollaries we now wish to obtain are derived by applying the theorem to the dual $G^\wedge$ of some group G, but to do this we need to ensure first that $G^\wedge$ is a k-space. This is so, for example, if G is locally compact (for then so is $G^\wedge$), or if G is hemicompact (for then $G^\wedge$ is first countable) [14].

Recall that a space X is a *k_ω-space* if it has the weak topology with respect to an increasing sequence $\{X_n\}$ of compact subsets with union X. Such a space is both hemicompact and a k-space. All compact spaces, locally compact σ-compact spaces, and countable CW-complexes are k_ω-spaces, and the free and free Abelian topological groups on a k_ω-space are again k_ω [11], [10]. The theorem and Corollary 1 now give

COROLLARY 2. *Suppose that the Abelian topological group G is reflexive. If $G^\wedge$ is a k-space then the path-component of the identity in G is the union of the one-parameter subgroups of G* (*and is thus divisible*); *and the same conclusion holds in particular if G is a k_ω-space.*

The next corollary reduces the study of the reflexivity of many locally path-connected groups to the study of those which are also path-connected.

COROLLARY 3. *Suppose that G is reflexive. If G is a locally path-connected k_ω-space, then it is topologically isomorphic to $P \oplus D$, where P is reflexive and is a path-connected, locally path-connected k_ω-space, and D is countable and discrete.*

PROOF. The path-component P of the identity in G is a (locally path-connected) open subgroup, and the work of [17] (see also Corollary 3.4 of [14]) therefore shows that P is reflexive. It is also a k_ω-space because it is closed in G, and thus Corollary 2 applies, showing that P is divisible. By (6.22)(b) of [4] any open divisible subgroup of G is a topological direct summand of G, whence $G \cong P \oplus D$, where D is the discrete group G/P; and this quotient, being thus a discrete k_ω-space, is countable.

COROLLARY 4. *Let X be a k_ω-space containing a nontrivial path. Then $A(X)$, the free Abelian topological group on X, is not reflexive.*

PROOF. The underlying abstract group of $A(X)$ is a free Abelian group, which clearly has no nontrivial divisible subgroups. As remarked earlier, $A(X)$ is a k_ω-space, so if it were reflexive, Corollary 2 would show it to be totally path-disconnected, contradicting our assumption on X. □

In [14] Noble asked: Is every complete Abelian k-group with sufficiently many characters reflexive? (Noble defines a k-group as a topological group on which every k-continuous homomorphism is continuous; the reader should note that there are several other quite different definitions in the literature. In addition, k-groups in Noble's sense need not be k-spaces, although any group which is a k-space (or a k_ω-space) is clearly a k-group.) Corollary 4 enables us to give a negative answer to Noble's question, once we note that for any k_ω-space X, $A(X)$ is complete and has sufficiently many characters to separate points. Completeness follows from Theorem 2 of [6] where it is proved that any group which is a k_ω-space is complete. On the other hand, if $w = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ is a nontrivial element of $A(X)$ (with $n \geq 1$, $a_1, a_2, \ldots, a_n$ nonzero integers and $x_1, x_2, \ldots, x_n$ distinct elements of X), choose $z \in T$ for which $z^{a_1} \neq 1$ and extend the map

$$x_1 \to z, \qquad x_i \to 1, \qquad i = 2, 3, \ldots, n,$$

to a continuous function $\phi: X \to T$. By definition of $A(X)$, ϕ then extends to a character $\Phi: A(X) \to T$, and $\Phi(w) = z^{a_1} \neq 1$ (cf. similar arguments in [3] and [12]).

It is well known that a character defined on a closed subgroup of a locally compact Abelian group can be extended to a character on the whole group. Without the assumption of local compactness, however, the situation is less clear, and it therefore seems worth noting that at least some of the groups discussed in Corollary 4 fail to have this property.

Specifically, consider the group $A(I^2)$, where I is the unit interval [0, 1]. The boundary $B = (\{0, 1\} \times I) \cup (I \times \{0, 1\})$ of I^2 is clearly homeomorphic to T–let $i: B \to T$ be a homeomorphism–and since it is closed in I^2 it generates a copy of $A(B)$ as a closed subgroup of $A(I^2)$ (cf. Theorem 3 of [10]). Furthermore, the character $i^*: A(B) \to T$ induced by i cannot be extended to $A(I^2)$, since such an extension would then induce, in effect, a retraction of I^2 onto its boundary, in contradiction of Brouwer's theorem [2, XVI, 2.1].

The proof of Theorem 4 of [13] shows that whenever a k_ω-space X contains a nontrivial path, $A(X)$ has a closed subgroup $A(I)$, so it would be of some interest to know whether an analogue of Theorem 1 of [13] holds for free Abelian topological groups, and, in particular, whether $A(I)$ contains a copy of $A(I^2)$. An affirmative answer would enable us to apply the above example to all the groups of Corollary 4.

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