AN EXTENSION OF AN OPERATOR INEQUALITY FOR S-NUMBERS

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Abstract. If it is assumed that the s-numbers associated with a given compact operator are ordered in the usual fashion, a basic result concerning infinite series of powers of these s-numbers can be appropriately restated so as to refer solely either to the lead terms of the series or to its tail. A simple proof, based upon an interesting auxiliary result concerning stochastic matrices, is given for this useful improvement.

A fundamental result in operator theory (see, for example, McCarthy [6], Gohberg and Krein [5], or Dunford and Schwartz [3]) relates infinite power sums of the s-numbers (characteristic numbers) of a given operator and corresponding power sums of the norms of the images, under the operator, of orthonormal basis vectors for the underlying Hilbert space. In the case of a compact operator $T$ acting on a Hilbert space $\mathcal{H}$, the following represents an extension of this result found useful in recent investigations of integral operators [2], [7]:

**Theorem.** If $0 < r < 2$, then for arbitrary $m > 1$,

$$\sum_{n=m}^{\infty} \left[ s_n(T) \right]^r = \inf \sum_{n=m}^{\infty} \| T\Psi_n \|^r,$$

while if $2 < r < \infty$,

$$\sum_{n=1}^{m} \left[ s_n(T) \right]^r = \sup \sum_{n=1}^{m} \| T\Psi_n \|^r.$$

In these expressions the s-numbers are ordered in their natural nonincreasing manner and the inf and sup are taken over all orthonormal bases of $\mathcal{H}$. If $r \neq 2$, equality occurs, as expected, if and only if $\{ \Psi_n \}$ is an appropriately ordered orthonormal basis for $\mathcal{H}$ consisting of characteristic functions (eigenvectors) for the related nonnegative definite operator $T^*T$.

The proof of (2) can be accomplished by employing several auxiliary results of Gohberg and Krein [5]. Alternatively, both (1) and (2) can be established making appropriate use of a theorem of Fan [4] (see also Beckenbach and Bellman [1, p. 77]). A simple and unencumbered demonstration, however, can be based directly upon the following result on doubly-stochastic matrices—a

Received by the editors May 11, 1976 and, in revised form, June 28, 1976.


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result which was subsidiary to the Fan theorem, but certainly is of interest in its own right.

**Lemma.** Let the nonnegative elements $p_{ij}$ ($i, j = 1, 2, \ldots$) be such that

$$\sum_{i=1}^{\infty} p_{ij} < 1 \quad (j = 1, 2, \ldots), \quad \sum_{j=1}^{\infty} p_{ij} < 1 \quad (i = 1, 2, \ldots).$$

If $a_i, b_i$ ($i = 1, 2, \ldots$) are two nonincreasing sequences of nonnegative numbers, then

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{n} p_{ij} b_j \leq \sum_{i=1}^{m} a_i b_i. \quad (3)$$

In order to prove the main theorem, we let $\{\Phi_n\}$ be a sequence of orthonormalized characteristic functions of the operator $T^*T$ corresponding to the naturally ordered $s$-numbers $s_n(T)$. Then for arbitrary $\Psi$ in $\mathcal{H}$ and $r > 2$,

$$\|T\Psi\|_r = \left( \sum_{j=1}^{\infty} \left[ s_j(T) \right]^r |(\Psi, \Phi_j)|^2 \right)^{r/2} \leq \sum_{j=1}^{\infty} \left[ s_j(T) \right]^r |(\Psi, \Phi_j)|^2 \left( \sum_{j=1}^{\infty} |(\Psi, \Phi_j)|^2 \right)^{(r-2)/2},$$

by Hölder’s inequality. If $\{\Psi_n\}$ is an orthonormal basis for $\mathcal{H}$, it follows, using Parseval’s relation, that

$$\|T\Psi_n\|_r < \sum_{n=1}^{m} \left[ s_n(T) \right]^r |(\Psi_n, \Phi_j)|^2.$$  

This inequality and the identifications $a_i = 1, b_j = [s_j(T)]', p_y = |(\Psi, \Phi_j)|^2$ in the Fan Lemma immediately lead to the desired result, namely

$$\sum_{n=1}^{m} \|T\Psi_n\|_r < \sum_{n=1}^{m} \left[ s_n(T) \right]'_{r}.$$  

In the case of $0 < r < 2$, the comparable chain of inequalities is

$$\sum_{n=m}^{\infty} \|T\Psi_n\|_r > \sum_{n=m}^{\infty} \sum_{j=1}^{\infty} \left[ s_j(T) \right]'_{r} |(\Psi_n, \Phi_j)|^2 \quad \left( \sum_{j=1}^{\infty} \left[ s_j(T) \right]'_{r} \right)^2 = \sum_{j=1}^{\infty} \left[ s_j(T) \right]'_{r} \left( 1 - \sum_{n=1}^{m-1} |(\Psi_n, \Phi_j)|^2 \right) \geq \sum_{n=m}^{\infty} \left[ s_n(T) \right]'_{r}.$$  

When $\Psi_n = \Phi_n$ ($n = 1, 2, \ldots$), all the various inequalities above become equalities—and when $r \neq 2$, only then—thereby completing the proof of (1), (2) and the Theorem.
REFERENCES


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