PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 65, Number 1, July 1977

AN EXTENSION OF AN OPERATOR INEQUALITY FOR s-NUMBERS

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ABSTRACT. If it is assumed that the s-numbers associated with a given compact operator are ordered in the usual fashion, a basic result concerning infinite series of powers of these s-numbers can be appropriately restated so as to refer solely either to the lead terms of the series or to its tail. A simple proof, based upon an interesting auxiliary result concerning stochastic matrices, is given for this useful improvement.

A fundamental result in operator theory (see, for example, McCarthy [6], Gohberg and Kreĭn [5], or Dunford and Schwartz [3]) relates infinite power sums of the s-numbers (characteristic numbers) of a given operator and corresponding power sums of the norms of the images, under the operator, of orthonormal basis vectors for the underlying Hilbert space. In the case of a compact operator T acting on a Hilbert space $\mathcal{H}$, the following represents an extension of this result found useful in recent investigations of integral operators [2], [7]:

THEOREM. *If* $0 < r \leqslant 2$, *then for arbitrary* $m \geqslant 1$,

$$\sum_{n=m}^{\infty} \left[s_n(T)\right]^r = \inf \sum_{n=m}^{\infty} \|T\Psi_n\|^r, \tag{1}$$

while if $2 \leqslant r < \infty$,

$$\sum_{n=1}^{m} \left[s_n(T)\right]^r = \sup \sum_{n=1}^{m} \|T\Psi_n\|^r. \tag{2}$$

In these expressions the s-numbers are ordered in their natural nonincreasing manner and the inf and sup are taken over all orthonormal bases of $\mathcal{H}$. If $r \neq 2$, equality occurs, as expected, if and only if $\{\Psi_n\}$ is an appropriately ordered orthonormal basis for $\mathcal{H}$ consisting of characteristic functions (eigenvectors) for the related nonnegative definite operator T^*T.

The proof of (2) can be accomplished by employing several auxiliary results of Gohberg and Kreĭn [5]. Alternatively, both (1) and (2) can be established making appropriate use of a theorem of Fan [4] (see also Beckenbach and Bellman [1, p. 77]). A simple and unencumbered demonstration, however, can be based directly upon the following result on doubly-stochastic matrices–a

Received by the editors May 11, 1976 and, in revised form, June 28, 1976.
AMS (*MOS*) *subject classifications* (1970). Primary 47B05, 47A10; Secondary 47B10.

result which was subsidiary to the Fan theorem, but certainly is of interest in its own right.

LEMMA. *Let the nonnegative elements* p_{ij} $(i, j = 1, 2, \dots)$ *be such that*

$$\sum_{i=1}^{\infty} p_{ij} \leqslant 1 \ (j = 1, 2, \dots,), \quad \sum_{j=1}^{\infty} p_{ij} \leqslant 1 \ (i = 1, 2, \dots,).$$

If a_i, b_i $(i = 1, 2, \dots)$ *are two nonincreasing sequences of nonnegative numbers, then*

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{n} p_{ij} b_j \leqslant \sum_{i=1}^{m} a_i b_i. \tag{3}$$

In order to prove the main theorem, we let $\{\Phi_n\}$ be a sequence of orthonormalized characteristic functions of the operator T^*T corresponding to the naturally ordered s-numbers $s_n^2(T)$. Then for arbitrary Ψ in $\mathcal{H}$ and $r \geqslant 2$,

$$\|T\Psi\|^r = \left\{ \sum_{j=1}^{\infty} [s_j(T)]^2 |(\Psi, \Phi_j)|^2 \right\}^{r/2}$$

$$\leqslant \sum_{j=1}^{\infty} [s_j(T)]^r |(\Psi, \Phi_j)|^2 \left\{ \sum_{j=1}^{\infty} |(\Psi, \Phi_j)|^2 \right\}^{(r-2)/2},$$

by Hölder's inequality. If $\{\Psi_n\}$ is an orthonormal basis for $\mathcal{H}$, it follows, using Parseval's relation, that

$$\sum_{n=1}^{m} \|T\Psi_n\|^r \leqslant \sum_{n=1}^{m} \sum_{j=1}^{\infty} [s_j(T)]^r |(\Psi_n, \Phi_j)|^2.$$

This inequality and the identifications $a_i = 1$, $b_j = [s_j(T)]^r$, and $p_{ij} = |(\Psi_i, \Phi_j)|^2$ in the Fan Lemma immediately lead to the desired result, namely

$$\sum_{n=1}^{m} \|T\Psi_n\|^r \leqslant \sum_{n=1}^{m} [s_n(T)]^r.$$

In the case of $0 < r \leqslant 2$, the comparable chain of inequalitities is

$$\sum_{n=m}^{\infty} \|T\Psi_n\|^r \geqslant \sum_{n=m}^{\infty} \sum_{j=1}^{\infty} [s_j(T)]^r |(\Psi_n, \Phi_j)|^2$$

$$= \sum_{j=1}^{\infty} [s_j(T)]^r \left[1 - \sum_{n=1}^{m-1} |(\Psi_n, \Phi_j)|^2 \right]$$

$$\geqslant \sum_{n=m}^{\infty} [s_n(T)]^r.$$

When $\Psi_n = \Phi_n$ $(n = 1, 2, \dots)$, all the various inequalities above become equalities–and when $r \neq 2$, only then–thereby completing the proof of (1), (2) and the Theorem.

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