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UNION OF CONVEX HILBERT CUBES

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ABSTRACT. We show that the finite union of Keller cubes in a Hilbert space is homeomorphic to the Hilbert cube provided every subcollection intersects in a Hilbert cube.

1. **Introduction.** In this paper, we establish the following result:

If $K_1, K_2, \ldots, K_n$ is a collection of convex Hilbert cubes contained in l_2 such that every subcollection intersects in a Hilbert cube, then $\bigcup_{i=1}^n K_i$ is homeomorphic to the Hilbert cube.

This result, for $n = 2$, answers a question stated in [**AK**, LS4, p. 166]. Because of its potential usefulness in studying hyperspaces of compact convex sets the same question was also posed in [**NQS$_2$**].

The more general question (see [**A$_1$**] and [**AB**]) of whether the union of two Hilbert cubes which intersect in a Hilbert cube is a Hilbert cube has been recently solved in the negative by R. Sher [**S**]. Results in a positive direction have been obtained by R. D. Anderson [**W$_1$**], N. Kroonenberg and R. Wong [**KW**] and M. Handel [**H$_2$**]. It should be mentioned that the results of this paper are easy consequences of the work in [**H$_2$**] and [**K**].

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2. **Some preliminaries.** The reader is referred to [**KU**] as a general reference for results and definitions in the area of topology. In what follows, we will denote by Q the Hilbert cube $Q = \prod_{i=1}^{\infty}[-1, 1]_i$. A Hilbert cube is a space homeomorphic ($\cong$) to Q. By a Keller cube we mean a Hilbert cube which is a compact convex subset of Hilbert space l_2. The reader should note that, since every compact subset of a metrizable locally convex topological vector space can be affinely embedded in l_2, the results of this paper will be true in such an "apparently" more general setting. A set $A \subset Q$ is said to be a *Z-set* if for any nonempty and homotopically trivial open subset U of Q, $U - A$ is nonempty and homotopically trivial. If Y is a metric space, a set $A \subset Y$ is said to be *k-Lcc* (k is a nonnegative integer) embedded in Y if for every $p \in A$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that any $f: S^k \to B(\delta, p) - A$ can

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be extended to a mapping $\bar{f}$: $B^{k+1} \to B(\varepsilon, p) - A$, where $B(\varepsilon, p) = \{y \in Y: d(y, p) < \varepsilon\}$. If $X \subset l_2$ then the *linear span of* X, denoted Sp(X), is the set of all $y \in l_2$ such that y is collinear with at least two distinct points of X. We will denote the closure of a set N by cl(N) and the convex hull of N by co(N). A metric space X is said to be LC^n if, for every $0 \leq k \leq n$, for every $p \in X$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that any mapping of S^k into $B(\delta, p)$ can be extended to a mapping of B^{k+1} into $B(\varepsilon, p)$.

A set $M = \bigcup_{i=1}^{\infty} M_i$ is said to be a (f.d.) CAP set for Q if, for each i, M_i is a (f.d.) Z-set in Q, $M_i \subset M_{i+1}$ and, given n_0, $\varepsilon > 0$ and the (f.d.) Z-set $Y \subset Q$, there exist an n_1 and a space homeomorphism h of Q onto Q such that $h|M_{n_0} = \text{id}$, $h(Y) \subset M_{n_1}$ and $d(h, \text{id}_Q) < \varepsilon$. The concept of a CAP set was introduced by R. D. Anderson (cf. [A_2]). (*Note.* f.d. is brief for "finite dimensional".)

In his paper [H_2] on sums of Hilbert cubes, Michael Handel has all but stated the following result:

LEMMA A (M. HANDEL). *Let* $Q_1 \cong Q_2 \cong Q_1 \cap Q_2 = Q_3 \cong Q$. *If* Q_2 *has an f.d. CAP set* $M = \bigcup_{i=1}^{\infty} M_i$ *such that* $M_i \cap Q_1$ *is a Z-set in* Q_1, *for each i, then* $Q_1 \cup Q_2 \cong Q$.

3. The main results.

(3.1) LEMMA. *If* $K = \bigcup_{i=1}^{n} K_i$ *is a Hilbert cube in* l_2 *such that, for each i,* K_i *is a Keller cube and such that* $\bigcap_{i=1}^{n} K_i \cong Q$, *then any closed set of finite dimensional linear span lying in K is a Z-set in K.*

PROOF. Let $P \subset K$ be a closed set of finite linear span. Nelly Kroonenberg has shown [K] that a finite dimensional closed subset of Q which is 1-*Lcc* embedded in Q is a Z-set in Q. We will show that P is 1-*Lcc* embedded in K. Let $p \in P$ and $\varepsilon > 0$ be given. Let $\delta > 0$ be chosen so that if K_j is such that $p \notin K_j$ then $B(\delta, p) \cap K_j = \emptyset$. Let f: $S^1 \to B^*(\delta, p) - P$, where $B^*(\delta, p) = B(\delta, p) \cap K$. We want to show there is a homotopy h: $S^1 \times I \to K$ such that $h(s, 0) = f(s)$, $h(S^1 \times 1)$ is a polyhedron and $h(S^1 \times t) \subset B^*(\delta, p) - P$ for all $0 \leq t \leq 1$. We will show that

(*) for every $\gamma > 0$, there exists a triangulation $T = \{v_1, v_2, \ldots, v_m\}$ of S^1 such that diam$(f(v_i, v_{i+1})) < \gamma/2$ and for each i there exists a K_j such that $\{f(v_i), f(v_{i+1})\} \subset K_j$.

Under these circumstances, we can conclude that any mapping g of S^1 onto $\bigcup_{i=1}^{m} [f(v_i), f(v_{i+1})]$ such that $g(v_i, v_{i+1}) = [f(v_i), f(v_{i+1})]$ and $g(v_i) = f(v_i)$ for $i = 1, 2, \ldots, m$ (note $v_{n+1} = v_1$) satisfies the property that $d(g, f) < \gamma$. Since $B^*(\delta, p)$ is (in particular) an LC^1 space, the above is enough to guarantee the existence of the desired homotopy h. To see that (*) holds, let $\gamma > 0$ be given and let $T = \{v_1, v_2, \ldots, v_m\}$ be a triangulation of S^1 such that diam$(f((v_i, v_{i+1}))) < \gamma/2$. If, for every i, $\{f(v_i), f(v_{i+1})\} \subset K_j$, for some j, we are done. Suppose there does not exist a K_j such that $\{f(v_i), f(v_{i+1})\} \subset K_j$. Let $u_1 = \sup\{u \in (v_i, v_{i+1})$: $f(u)$ and $f(v_i)$ lie in some common $K_j\}$. Note

$u_1 \neq v_i$. If $f(u_1)$ and $f(v_{i+1})$ belong to K_j for some j, then we merely refine T by the addition of the vertex u_1 between v_i and v_{i+1}. If $f(u_1)$ and $f(v_{i+1})$ do not belong to a common K_j, then let $u_2 = \sup\{u: u \in (u_1, v_{i+1})$ and $f(u)$ and $f(u_1)$ belong to a common $K_j\}$. Proceeding in this fashion, we eventually obtain $u_1, u_2, \ldots, u_s$ such that, for $k < s$, $f(u_k)$ and $f(v_{i+1})$ do not belong to a common K_j, $\{f(u_{k-1}), f(u_k)\}$ is contained in some K_j, $f(u_i)$ and $f(u_{k+1})$ do not belong to a common K_j, for $1 \leqslant i \leqslant k-1$, and $f(u_s)$ and $f(v_{i+1})$ belong to a common K_j. We now refine the triangulation T by the addition of the vertices $u_1, u_2, \ldots, u_s$ between v_i and v_{i+1}. We can now clearly conclude the validity of (*). So, let $h: S^1 \times I \to K$ be a homotopy such that $h(s, 0) = f(x)$, $h(S^1 \times 1)$ is a polyhedron, and

$$h(S^1 \times I) \subset B^*(\delta, p) - P.$$

Let $g: S^1 \times I \to B(\delta, p)$ be defined by $g(s, t) = (1-t)h(s, 1) + tp$. Let $q \in \bigcap_{i=1}^n K_i$ be such that $q \notin \operatorname{Sp}(g(S^1 \times I) \cup P)$. Let $l > 0$ be chosen so that, for $r \in [0, l]$ and $(s, t) \in S^1 \times I$, $(1-r)g(s, t) + rq \in B^*(\delta, p)$. Define $H: S^1 \times [0, l+2] \to B^*(\delta, p) - P$ by

$$H(s,t) = \begin{cases} h(s,t) & \text{for } t \in [0,1], \\ (2-t)g(s,0) + (t-1)q & \text{for } t \in [1, 1+l], \\ (1-l)g(s, t-(1+l)) + lq & \text{for } t \in [1+l, 2+l]. \end{cases}$$

The lemma is proved.

(3.2) THEOREM. *Let $K_1, K_2, \ldots, K_n$ be a collection of Keller cubes such that every subcollection of the $\{K_i\}_{i=1}^n$ intersects in a Hilbert cube. Then $\bigcup_{i=1}^n K_i \cong Q$.*

PROOF. The result is true for $n = 1$. Assume it is true for $k \leqslant n$ and suppose $K_1, K_2, \ldots, K_n, K_{n+1}$ is a collection of Keller cubes such that the hypothesis of the lemma is satisfied. Let $M = \bigcup_{i=1}^\infty M_i$ be an f.d. CAP set for K_{n+1} such that M_i is convex for each i. That such an M can be found follows from Proposition 5.1 of [**BP**, p. 162]. By the induction hypothesis, $\bigcup_{i=1}^n K_i \cong Q$ and $K_{n+1} \cap (\bigcup_{i=1}^n K_i) = \bigcup_{i=1}^n (K_{n+1} \cap K_i) \cong Q$. By Handel's result (Lemma A) it suffices to see that $M_i \cap (\bigcup_{j=1}^n K_j)$ is a Z-set in $\bigcup_{j=1}^n K_j$. But, $M_i \cap (\bigcup_{j=1}^n K_j)$ is a finite sum of finite dimensional convex sets. By (3.1), we have that $M_i \cap (\bigcup_{i=1}^n K_i)$ is a Z-set in $\bigcup_{i=1}^n K_i$. The lemma is proved.

(3.3) COROLLARY. *Let $K_1, K_2, \ldots, K_n$ be a collection of Keller cubes such that if $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ then either $\bigcap_{j=1}^k K_{i_j} = \varnothing$ or $\bigcap_{j=1}^k K_{i_j} \cong Q$. Then, $\bigcup_{i=1}^n K_i$ is a Q-manifold.*

PROOF. Let $q \in \bigcup_{i=1}^n K_i$ and let $i_1, i_2, \ldots, i_k$ be the maximal set of indices such that $q \in \bigcap_{j=1}^k K_{i_j}$. Then $\bigcup_{j=1}^k K_{i_j}$ is a neighborhood of q and $Q \cong \bigcup_{j=1}^k K_{i_j}$ by (3.2). The theorem is proved.

T. A. Chapman has proved (see [$\mathbf{C_1}$] or [$\mathbf{C_2}$]) that a contractible compact Q-manifold is homeomorphic to Q. We thus have the following

(3.4) COROLLARY. *Let* $K_1, K_2, \ldots, K_n$ *be a collection of Keller cubes such that if* $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ *then either* $\bigcap_{j=1}^{k} K_{i_j} = \varnothing$ *or* $\bigcap_{j=1}^{k} K_{i_j} \cong Q$. *Then* $\bigcup_{i=1}^{n} K_i \cong Q$ *if and only if* $\bigcup_{i=1}^{n} K_i$ *is contractible.*

This next corollary is a restatement of (3.2) for $n = 2$.

(3.5) COROLLARY. *If* $K_1 \cong K_2 \cong K_1 \cap K_2 \cong Q$ *are Keller cubes then* $K_1 \cup K_2 \cong Q$.

(3.6) REMARK. If P is a polyhedron in E^n, then we can view $P \times Q$ as a subset of $E^n \times l_2$. As such, $P \times Q$ is clearly a finite union of convex Hilbert cubes which have the proper intersection properties. Recalling that the results of this paper hold just as well in this setting, we have by (3.4) that $P \times Q$ is locally homeomorphic to Q. This result can now be used to give another proof of the result of J. E. West (see Corollary 5.2 of [$\mathbf{W_2}$]), which states that if X is a space which can be triangulated by a locally finite simplicial complex, then the product of X with Q is locally homeomorphic to Q.

4. **Some discussion.** It is reasonable to ask whether the results of the preceding section can be extended to cover situations with less stringent intersection requirements than those of (3.2) or (3.3). In this section we will give a counterexample. In fact, we will give three examples. The first example shows that (in a nontrivial sense) at least sometimes the union of three Keller cubes can be homeomorphic to Q even when the intersection of two of them is just a 2-cell. The second example is a nontrivial example of four Keller cubes which pairwise intersect in Keller cubes, whose union is homeomorphic to Q but whose total intersection is a 2-cell. The last example is of three Keller cubes which pairwise intersect in Keller cubes and whose total intersection is a 2-cell (or n-cell), and the union of the Keller cubes is not homeomorphic to Q. The argument of the last example is suggested to us by the referee to whom we acknowledge our thanks.

If X is a subset of a Banach space then by cc(X) (the cc-*hyperspace* of X) is meant the space of compact convex subsets of X with the Hausdorff metric (see [$\mathbf{NQS_1}$]). Using the techniques of §2 of [$\mathbf{NQS_3}$] it can be seen that if $\{X_i: i = 1, 2, \ldots, n\}$ is a collection of compact subsets of a Banach space then $\bigcup_{i=1}^{n} \mathrm{cc}(X_i)$ can be affinely embedded into l_2. All of the examples which follow are obtained by taking unions of cc-hyperspaces of compact convex subsets of R^3.

(4.1) EXAMPLE. In R^2 let

$Y_1 = \mathrm{co}(\{(0, 0), (0, 1), (-1, 1), (-1, 0)\})$,

$Y_2 = \mathrm{co}(\{(0, 0), (-2, 0)\})$, and

$Y_3 = \mathrm{co}(\{(0, 0), (0, 2)\})$.

In R^3 let $X_i = Y_i \times [0, 1]$ and let $K_i = \mathrm{cc}(X_i)$ for $i = 1, 2, 3$. It follows from Theorem (2.2) of [$\mathbf{NQS_3}$] (see also [$\mathbf{NQS_1}$]) that $K_1, K_2, K_3, K_1 \cap K_2$ and $K_1 \cap K_3$ are all homeomorphic to Q. Since $K_2 \cap K_3 = K_1 \cap K_2 \cap K_3 = \mathrm{cc}(X_1 \cap X_2 \cap X_3)$ is the cc-hyperspace of an arc, we have that $K_2 \cap K_3 = K_1$

$\cap K_2 \cap K_3$ is a 2-cell. We wish to show that $K = \bigcup_{i=1}^3 K_i \cong Q$. But $K_1 \cap K_2 \cong Q$ and K_1, K_2 are Keller cubes. Thus, $K_1 \cup K_2 \cong Q$ by (3.5). Since $K_3 \cap (K_1 \cup K_2) = K_3 \cap K_1 \cong Q$ since $K_1 \cap K_3$ is a Z-set in $K_1 \cup K_2$ (see Example (4.6) of **[NQS₃]**) we have, by Theorem 1 of **[H₂]**, that $K_3 \cup (K_1 \cup K_2) \cong Q$. We have shown that $\bigcup_{i=1}^3 K_i \cong Q$.

A trivial example of four Keller cubes which pairwise intersect in Keller cubes, have total intersection a 2-cell and whose union is homeomorphic to Q can be obtained from any example of three Keller cubes with the same intersection properties by taking the fourth Keller cube to be one which contains the union of the other three. We prefer to have an example for which no Keller cube contains any of the others.

(4.2) Example. In R^1 let

$Y_1 = \mathrm{co}(\{(0, 0), (0, 1), (-1, 1), (-1, 0)\})$,

$Y_2 = \mathrm{co}(\{(0, 0), (1, 0), (1, 1), (0, 1)\})$,

$Y_3 = \mathrm{co}(\{(-1, \frac{1}{2}), (-1, 0), (1, 0), (1, \frac{1}{2})\})$, and

$Y_4 = \mathrm{co}(\{(-1, 0), (-1, -1), (1, -1), (1, 0)\})$.

In R^3 let $X_i = Y_i \times [0, 1]$ and let $K_i = \mathrm{cc}(Y_i)$ for $i = 1, 2, 3, 4$. It follows from Theorem (2.2) of **[NQS₃]** that $K_1, K_2, K_3, K_4, K_1 \cap K_2, K_1 \cap K_3, K_1 \cap K_4, K_2 \cap K_3, K_2 \cap K_4$ and $K_3 \cap K_4$ are all homeomorphic to Q. As in (4.1), $\bigcap_{i=1}^4 K_i$ is the cc-hyperspace of an arc and is thus a 2-cell. To see that $\bigcup_{i=1}^4 K_i \cong Q$, we will first consider $\bigcup_{i=1}^3 K_i$. Since $\bigcap_{i=1}^3 K_i \cong Q$, we have by (3.2) that $\bigcup_{i=1}^3 K_i \cong Q$. Now, $K_4 \cap (\bigcup_{i=1}^3 K_i) = K_4 \cap K_3 \cong Q$ and $K_4 \cap (\bigcup_{i=1}^3 K_i)$ is a Z-set in K_4 (see Example (4.6) of **[NQS₃]**). We thus have that $K_4 \cup (\bigcup_{i=1}^3 K_i) = \bigcup_{i=1}^4 K_i \cong Q$.

This next example is of three Keller cubes which pairwise intersect in Keller cubes and have total intersection of a 2-cell. Their union, however, is not homeomorphic to Q.

(4.3) Example. In R^2 let

$Y_1 = \mathrm{co}(\{(0, 0), (1, 0), (2, -1), (0, -1)\})$,

$Y_2 = \mathrm{co}(\{(0, 0), (1, 1), (2, 1), (0, 1)\})$, and

$Y_3 = \mathrm{co}(\{(1, 0), (2, 1), (2, -1)\})$.

In R^3 let $X_i = Y_i \times [0, 1]$ for $i = 1, 2, 3$. Let $K_i = \mathrm{cc}(X_i)$ for $i = 1, 2, 3$. We have that $K_{i_1} \cap K_{i_2} \cong Q$ for $i_1 \neq i_2$ and $\bigcap_{i=1}^3 K_i \cong I^2$. It can be easily shown, using Anderson's Z-set homeomorphic extension theorem, that $K_1 \cup K_2 \cup K_3$ is homeomorphic to $\mathrm{Cone}(S^1 \times Q) \times I^2$, which is not homeomorphic to Q. We argue as follows: Let $(v_1, v_2) \in \mathrm{Cone}(S^1 \times Q) \times I^2$, where v_1 is the cone point and v_2 is any interior point of I^2. If $\mathrm{Cone}(S^1 \times Q) \times I^2 \cong Q$, then $\mathrm{Cone}(S^1 \times Q) \times I^2 \setminus \{(v_1, v_2)\}$ is contractible. But

$$\begin{aligned}\mathrm{Cone}(S^1 \times Q) \times I^2 \setminus \{(v_1, v_2)\}\ &(\text{homotopic})\\ &\sim \mathrm{Cone}(S^1) \times I^2 \setminus \{(v_1, v_2)\}\\ &\sim I^4 \setminus \{\text{an interior point}\} \sim S^3,\end{aligned}$$

which is not contractible.

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