

HANKEL OPERATORS WITH DISCONTINUOUS SYMBOL

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ABSTRACT. Douglas's localisation theory for Toeplitz operators is used to show that there exist certain Hankel operators with discontinuous symbols which do not lie in the C^* -algebra generated by the Toeplitz operators. As a simple corollary we also see that these operators do not lie in the closed linear span of the positive Hankel operators.

Introduction. Let L^2 be the Hilbert space of square integrable functions on the unit circle T , with complete orthonormal basis $\{e_n; n \in \mathbb{Z}\}$ where e_n denotes the function z^n . J is the unitary on L^2 defined by $Je_n = e_{-n}$ for n in \mathbb{Z} . H^2 denotes the Hardy subspace spanned by $\{e_n; n \geq 0\}$ and P is the orthogonal projection of L^2 onto H^2 .

For an essentially bounded measurable function ϕ in L^∞ , the Toeplitz operator T_ϕ on H^2 , is defined by $T_\phi = PM_\phi|_{H^2}$ where M_ϕ is the usual multiplication operator on L^2 . We call ϕ the symbol of the Toeplitz operator T_ϕ . The Hankel operator on H^2 , with symbol ϕ in L^∞ , is defined by $S_\phi = PJM_\phi|_{H^2}$.

Whilst the Toeplitz and Hankel operators are formally rather similar, quite different techniques are usually needed to study them. However in this note we shall use Douglas's localisation theory for Toeplitz operators [2] to show that the closed linear span of the positive Hankel operators does not contain certain Hankel operators with discontinuous symbol. This result has also appeared in [4]. The proof was based on Widom's characterisation of positive Hankel operators [5] which we avoid here.

Notation. For ϕ in L^∞ , the functions ϕ^* and $\tilde{\phi}$ are defined by

$$\phi^*(z) = \phi(\bar{z}) \quad \text{and} \quad \tilde{\phi}(z) = \overline{\phi^*(z)} \quad (z \in T).$$

Thus we have $JM_\phi = M_{\phi^*}J$.

Let \mathbf{T} denote the C^* -algebra generated by the Toeplitz operators and let \mathbf{P} denote the closed linear span of the positive Hankel operators.

The following simple lemma shows that the symbol of the adjoint of a

Received by the editors November 8, 1976.

AMS (MOS) subject classifications (1970). Primary 47B35, 47C10.

Key words and phrases. Hankel operator, discontinuous symbol, Toeplitz operator, C^* -algebra, positive Hankel operator.

¹ This research was supported by the Science Research Council.

Hankel operator is not the 'local adjoint' of its symbol. The proof of Theorem 8 rests on this fact.

LEMMA 1. $S_\phi^* = S_{\bar{\phi}}$ for $\phi \in L^\infty$.

PROOF. For f and g in H^2 we have

$$(S_\phi f, g) = (JM_\phi f, g) = (M_{\phi^*} Jf, g) = (Jf, M_{\bar{\phi}} g) = (f, S_{\bar{\phi}} g).$$

LEMMA 2. $S_\phi^* S_\psi = T_{\bar{\phi}\psi} - T_{\bar{\phi}} T_{z\psi}$ for $\phi, \psi \in L^\infty$.

PROOF.

$$S_\phi^* S_\psi = PJM_{\bar{\phi}} PJM_\psi |H^2 = PM_{\bar{\phi}} JPJM_\psi |H^2 = PM_{\bar{\phi}z} (M_z JPJM_z) M_{z\psi} |H^2.$$

Since $M_z JPJM_z = I - P$ we have

$$S_\phi^* S_\psi = PM_{\bar{\phi}z} (I - P) M_{z\psi} |H^2 = T_{\bar{\phi}\psi} - T_{\bar{\phi}} T_{z\psi}.$$

LEMMA 3. $\mathbf{P} \subset \mathbf{T}$.

PROOF. Let S be a positive Hankel operator. Lemma 2 shows that $S^2 \in \mathbf{T}$. Since S is the unique positive square root of S^2 and \mathbf{T} is a C^* -algebra we have $S \in \mathbf{T}$ and the lemma follows.

A well-known classical theorem for Hankel operators is due to Hartman (see [3] or [1]). This states that a Hankel operator is compact if and only if its symbol can be chosen to be continuous. Since $S_\phi = 0$ only when $\phi \in zH^\infty$ it follows that S_ϕ is compact if and only if $\phi \in H^\infty + C(T)$. We shall use this result in Theorem 8.

LEMMA 4. If $\phi, \psi \in L^\infty$ and if ϕ or ψ is continuous then $T_{\phi\psi} - T_\phi T_\psi$ is compact.

PROOF. Lemma 2 shows that $S_{z\bar{\phi}}^* S_{z\psi} = T_{\phi\psi} - T_\phi T_\psi$ and Hartman's theorem completes the proof. Alternatively see [2, p. 184].

The following two localisation theorems appear in Douglas [2, pp. 198, 199].

For $\lambda \in T$, let J_λ be the closed ideal in \mathbf{T} generated by $\{T_\phi: \phi \in C(T), \phi(\lambda) = 0\}$ and let Φ_λ be the natural $*$ -homomorphism of \mathbf{T} onto the quotient C^* -algebra $\mathbf{T}_\lambda = \mathbf{T}/J_\lambda$. Let Φ be the $*$ -homomorphism from \mathbf{T} to $\sum_{\lambda \in T} \oplus \mathbf{T}_\lambda$ defined by $\Phi = \sum_{\lambda \in T} \oplus \Phi_\lambda$.

THEOREM 5. $\ker \Phi$ contains the ideal of compact operators K and Φ induces a $*$ -isometrical isomorphism

$$\Phi_c: \mathbf{T}/K \rightarrow \sum_{\lambda \in T} \oplus \mathbf{T}_\lambda.$$

Let Λ be the maximal ideal space of L^∞ and let $F_\lambda = \{m \in \Lambda; m(z) = \lambda\}$ be the fibre in Λ over λ for $\lambda \in T$.

THEOREM 6. *If $\{\phi_{ij}\}_{i,j=1}^n$ are functions in L^∞ with Gelfand transforms $\hat{\phi}_{ij}$ on Λ and $\lambda \in T$ then $\Phi_\lambda(\sum_i \prod_j T_{\phi_{ij}})$ depends only on the functions $\{\hat{\phi}_{ij}|F_\lambda\}_{i,j=1}^n$.*

COROLLARY 7. *If $\phi, \psi \in L^\infty$ and ψ is continuous on a neighborhood of $\lambda \in T$ then $\Phi_\lambda(T_{\phi\psi} - T_\phi T_\psi) = 0$.*

PROOF. Theorem 6 implies that we can assume ψ to be continuous on T . The corollary then follows from Lemma 4 and Theorem 5.

THEOREM 8. *Let α be a nonreal complex number of modulus one and let ϕ be a function in L^∞ which is continuous on T apart from a (proper) jump discontinuity at α . Then $S_\phi \notin T$.*

PROOF. By Lemma 2, $S_\phi^* S_\phi = T_{|\phi|^2} - T_{\bar{\phi}} T_\phi$ and so by Corollary 7 $\Phi_\lambda(S_\phi^* S_\phi) = 0$ for $\lambda \neq \alpha$.

Suppose now that $S_\phi \in T$. Since Φ_λ is a *-homomorphism $\Phi_\lambda(S_\phi)^* \Phi_\lambda(S_\phi) = 0$ for $\lambda \neq \alpha$. Similarly, since by Lemma 1 $S_\phi^* = S_{\bar{\phi}}$ and $\bar{\phi}$ is continuous apart from a jump discontinuity at $\bar{\alpha}$, it follows that $\Phi_\lambda(S_\phi) \Phi_\lambda(S_\phi)^* = 0$ for $\lambda \neq \bar{\alpha}$. Thus $\Phi_\lambda(S_\phi) = 0$ for all $\lambda \in T$. By Theorem 5 S_ϕ is compact and so by Hartman's theorem $\phi \in H^\infty + C(T)$. Since such functions cannot have jump discontinuities we conclude that $S_\phi \notin T$.

COROLLARY 9. *Let ϕ be a function in L^∞ which is continuous apart from a jump discontinuity at a nonreal point of the unit circle. Then S_ϕ does not lie in P .*

PROOF. Immediate from Theorem 8 and Lemma 3.

The corollary does not remain true if the discontinuities occur at -1 or $+1$. In fact in [4] it is shown that if ϕ is continuous apart from jump discontinuities at $+1$ or -1 then $S_\phi \in P$.

It would be interesting to settle whether T contains any other Hankel operators besides those in P .

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