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THE THOM SPACE PERIODICITY OF CLASSIFYING SPACES[1]

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ABSTRACT. If G is any topological group then there exists a classifying space B_G. In this paper we shall exhibit a fiber bundle ω over B_G such that the Thom complex B_G^ω is homeomorphic to B_G. As an application we give a new proof of the Freudenthal Suspension Theorem.

1. **Introduction.** As a starting point for this paper consider the sequence of complex projective spaces CP^k together with their canonical complex line bundles ω_k. These bundles are compatible with respect to the usual inclusions $CP^m \rightarrowtail CP^k$, and moreover there exist homeomorphisms $(CP^k)^{\omega_k} \cong CP^{k+1}$ also compatible with inclusions [1] (here X^ω denotes the Thom complex of ω over X). Taking the limit as $k \to \infty$ we obtain a homeomorphism $(CP^\infty)^\omega \cong CP^\infty$, where ω is the universal complex line bundle. This homeomorphism reveals a geometrically periodic structure on CP^∞ since the Thom complex $(CP^\infty)^\omega$ is like CP^∞ with a shift in dimensions.

For another example of periodicity consider the sequence of quaternionic projective spaces HP^k together with the associated quaternionic line bundles ω_k. Then again there exist homeomorphisms $(HP^k)^{\omega_k} \cong HP^{k+1}$ compatible with inclusions $HP^m \rightarrowtail HP^k$, and they produce in the limit a homeomorphism $(HP^\infty)^\omega \cong HP^\infty$, where ω is now the universal quaternionic line bundle [1]. Therefore HP^∞ also has a geometrically periodic structure.

These two examples tend to suggest that any classifying space B_G, where G is any topological group, should carry some kind of periodic structure. In fact, in §2 we shall construct a fiber bundle pair ω:

$$(CG, G) \rightarrowtail (D(\omega), S(\omega)) \to B_G,$$

where C denotes the unreduced cone functor, with the property that the quotient complex $D(\omega)/S(\omega)$ is naturally homeomorphic to B_G. To be explicit, the bundle ω is the (CG, G) bundle associated to Milnor's universal principal G bundle $E_G \to B_G$ [3] by the obvious G action on (CG, G). Then $D(\omega) - S(\omega)$ is the $CG - G$ bundle associated to the action of G on the open cone $CG - G$. Thus we consider the pair $(D(\omega), S(\omega))$ as a generalization of

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the disc bundle-sphere bundle pair of a vector bundle. Accordingly we shall define the Thom complex B_G^ω to be $D(\omega)/S(\omega)$.

In §3 we shall exploit the relative Serre spectral sequence of the bundle pair ω to give a type of algebraic periodicity for $H_*(B_G)$ and $H^*(B_G)$. One particular consequence of this spectral sequence will be a new proof of the Freudenthal Suspension Theorem.

2. **Universal bundles and periodicity.** Throughout this section G shall denote a completely arbitrary topological group. In order to construct the bundle ω over B_G we recall the join construction of Milnor [3]. Thus we let G act on the k-fold joins $G^k = G * \cdots * G$ by

$$G^k \times G \to G^k, \qquad (t_1 g_1 + \cdots + t_k g_k, g) \to t_1 g_1 g + \cdots + t_k g_k g.$$

Then by taking the quotient spaces $X_k = G^k/G$ we obtain for each $k \geqslant 1$ a principal G bundle $p_k\colon G^k \to X_k$. These bundles are obviously compatible with the inclusions $G^m \to G^k$ and so yield in the limit a universal principal G bundle $p_\infty\colon G^\infty \to X_\infty$. Following standard terminology we shall write E_G for G^∞ and B_G for X_∞. Thus we have a commutative diagram of principal bundles

$$\text{(2.1)}\qquad \begin{array}{ccccccccccc} G = G^1 & \rightarrowtail & \cdots & \rightarrowtail & G^k & \rightarrowtail & G^{k+1} & \rightarrowtail & \cdots & \rightarrowtail & E_G \\ \downarrow p_1 & & & & \downarrow p_k & & \downarrow p_{k+1} & & & & \downarrow p_\infty \\ pt = X_1 & \rightarrowtail & \cdots & \rightarrowtail & X_k & \rightarrowtail & X_{k+1} & \rightarrowtail & \cdots & \rightarrowtail & B_G \end{array}$$

To any of the bundles $p_k\colon G^k \to X_k$, $1 \leqslant k \leqslant \infty$, we now associate the fiber bundle pair ω_k whose total space is given by

$$(D(\omega_k), S(\omega_k)) = G^k \underset{G}{\times} (CG, G)$$

and whose projection $(D(\omega_k), S(\omega_k)) \to X_k$ is induced by $p_k\colon G^k \to X_k$. Then we clearly have $\omega_k | X_m = \omega_m$ for $m \leqslant k$ and so we shall usually drop the subscript from the notation. In particular we have constructed the fiber bundle pair ω:

$$\text{(2.2)}\qquad \begin{gathered} (CG, G) \overset{i}{\rightarrowtail} (D(\omega), S(\omega)) \to B_G, \\ \text{where } (D(\omega), S(\omega)) = (E_G \times (CG, G))/G. \end{gathered}$$

Notice that $(D(\omega_1), S(\omega_1)) = G \times_G (CG, G)$ may be naturally identified with (CG, G) by the map

$$G \underset{G}{\times} CG \to CG, \quad [g, [t, g']] \to [t, g'g^{-1}].$$

We adopt the notation that square brackets shall always denote equivalence classes. Since X_1 is a natural base point for B_G we take the inclusion $(D(\omega_1), S(\omega_1)) \rightarrowtail (D(\omega), S(\omega))$ to be the fiber map $(CG, G) \rightarrowtail^i (D(\omega), S(\omega))$ of (2.2).

(2.3) THE GEOMETRIC PERIODICITY THEOREM. *For each k, $1 \leqslant k \leqslant \infty$, the Thom complex X_k^ω is naturally homeomorphic to X_{k+1}. In particular there exists a homeomorphism $B_G^\omega \cong B_G$ which is functorial in G.*

PROOF. First consider the case $k < \infty$. A typical point of $D(\omega_k) = G^k \times_G CG$ is $[t_1 g_1 + \cdots + t_k g_k, [t, g]]$ and so we can define a map Φ': $D(\omega_k) \to X_{k+1}$ by

$$\Phi'[t_1 g_1 + \cdots + t_k g_k, [t, g]] = [(1 - t) g + t t_1 g_1 + \cdots + t t_k g_k].$$

The subspace of $D(\omega_k)$ given by setting $t = 0$ is mapped to a point. But this subspace is just $S(\omega_k)$ and so Φ' induces a map Φ: $D(\omega_k)/S(\omega_k) \to X_{k+1}$.

To construct an inverse of Φ consider the set A of all points $[u_1 h_1 + \cdots + u_{k+1} h_{k+1}] \in X_{k+1}$ such that $0 \leqslant u_1 < 1$. In other words $A = X_{k+1} - X_1 = X_{k+1} - pt$. Also A is the image set $\Phi'(D(\omega_k) - S(\omega_k))$ and we can define a map Ψ': $A \to D(\omega_k) - S(\omega_k)$ by

$$\Psi'[u_1 h_1 + \cdots + u_{k+1} h_{k+1}] = [t_1 g_1 + \cdots + t_k g_k, [t, g]]$$
$$\text{where } t_i g_i = (u_{i+1}/(1 - u_1)) h_{i+1}, t = 1 - u_1, g = h_1.$$

Then Ψ' extends to a map Ψ: $X_{k+1} \to D(\omega_k)/S(\omega_k)$ and one can easily check that Ψ is the inverse of Φ. The homeomorphisms $X_k^{\omega_k} \cong X_{k+1}$ so constructed are compatible with respect to the inclusions $X_k \rightarrowtail X_m$, and so produce in the limit a homeomorphism $X_\infty^\omega \cong X_\infty$. This proves (2.3).

The fiber map i: $(CG, G) \rightarrowtail (D(\omega), S(\omega))$ and the homeomorphism Φ: $D(\omega)/S(\omega) \cong B_G$ determine a map ϕ: $\Sigma G \to B_G$, where Σ is the unreduced suspension functor.

(2.4) LEMMA. *ϕ is the standard inclusion of X_2 into B_G.*

By applying the relative Serre spectral sequence to the fiber bundle pair (2.2) and using the homeomorphism $D(\omega)/S(\omega) \cong B_G$ of (2.3) we can derive

(2.5) THE ALGEBRAIC PERIODICITY THEOREM. *Suppose G is a connected topological group. Then there exist spectral sequences which are functorial in G.*

$$E_{s,t}^2 \cong H_s(B_G; \tilde{H}_t(\Sigma G)) \Rightarrow \tilde{H}_{s+t}(B_G),$$
$$E_2^{s,t} \cong H^s(B_G; \tilde{H}^t(\Sigma G)) \Rightarrow \tilde{H}^{s+t}(B_G).$$

If G is either S^1 or S^3 then ω is either the universal complex line bundle or the universal quaternionic line bundle and (2.3) reduces to the examples at the beginning of this paper. Moreover the spectral sequences of (2.5) collapse totally giving isomorphisms

$$H_s(B_G) \cong \tilde{H}_{s+n}(B_G), \quad H^s(B_G) \cong \tilde{H}^{s+n}(B_G) \quad \text{for all } s,$$
$$\text{where } n = 2 \text{ (resp. 4)} \quad \text{if } G = S^1 \text{ (resp. } S^3).$$

The periodicity theorems can be generalized by starting with an arbitrary principal G bundle p: $W \to X$ rather than with the special bundle $G \to pt$. The group G acts on the k-fold joins W^k producing principal G bundles p_k: $W^k \to X_k$ for all $k \geqslant 1$. Taking the limit we have a universal principal G bundle p_∞: $W^\infty \to X_\infty$ and a commutative diagram of principal G bundles

$$(2.6)\qquad \begin{array}{ccccccccccc} W = W^1 & \rightarrowtail & \cdots & \rightarrowtail & W^k & \rightarrowtail & W^{k+1} & \rightarrowtail & \cdots & \rightarrowtail & W^\infty \\ \downarrow p_1 & & & & \downarrow p_k & & \downarrow p_{k+1} & & & & \downarrow p_\infty \\ X = X_1 & \rightarrowtail & \cdots & \rightarrowtail & X_k & \rightarrowtail & X_{k+1} & \rightarrowtail & \cdots & \rightarrowtail & X_\infty \end{array}$$

Then for $1 \leqslant k \leqslant \infty$ we can construct a bundle pair ω_k over X_k,

$$(CW, W) \rightarrowtail (D(\omega_k), S(\omega_k)) \to X_k$$

by defining $(D(\omega_k), S(\omega_k)) = W^k \times_G (CW, W)$ and taking for the projection the map induced by p_k: $W^k \to X_k$. The proof of (2.3) now yields a homeomorphism $X_k^{\omega_k} \cong X_{k+1}/X_1$ compatible with inclusions. Thus we have proved

(2.7) THEOREM. *Suppose p: $W \to X$ is a principal G bundle. Then there exists a fiber bundle pair ω:*

$$(CW, W) \rightarrowtail (D(\omega), S(\omega)) \to X_\infty$$

and a periodicity homeomorphism $X_\infty^\omega \cong X_\infty/X_1$. Moreover the construction of ω and the periodicity are functorial for maps between principal bundles.

Algebraic periodicity then follows from the relative Serre spectral sequence.

(2.8) THEOREM. *Suppose p: $W \to X$ is a principal G bundle with W connected. Then there are spectral sequences*

$$E^2_{s,t} \cong H_s(X_\infty; H_t(CW, W)) \Rightarrow H_{s+t}(X_\infty, X_1),$$

$$E_2^{s,t} \cong H^s\big(X_\infty; H^t(CW, W)\big) \Rightarrow H^{s+t}(X_\infty, X_1),$$

which are functorial for maps between principal bundles.

In [4], the periodicity theorems, (2.7) and (2.8), were proved for the case where p: $W \to X$ is a regular covering space with deck transformation group G. The essential idea of the proof in that paper was to define and construct normal bundles ω_k for the embeddings $X_k \rightarrowtail X_{k+1}$. That approach carries over to this paper to give another proof of (2.7).

3. **Applications**. As an application of the homology spectral sequence in (2.5) we shall prove the Freudenthal Suspension Theorem. To do this we need the simplicial loop space construction of Milnor [2]. If $G(X)$ is the space of simplicial loops and $E(X)$ is the space of simplicial paths then $G(X)$ is a topological group and there exists a principal universal $G(X)$ bundle $E(X) \to X$. This bundle is equivalent to the Serre path-loop fibration $P(X) \to X$ since the obvious inclusion $E(X) \rightarrowtail P(X)$ is compatible with respect to projections onto X and produces a homotopy equivalence $G(X) \rightarrowtail \Omega(X)$.

Taking G to be $G(X)$ in (2.2) and (2.3) we see that we have a fiber bundle pair ω:

$$(3.1)\qquad (CG(X), G(X)) \overset{i}{\rightarrowtail} (D(\omega), S(\omega)) \to B_{G(X)}$$

such that $D(\omega)/S(\omega) \cong B_{G(X)}$. Since $G(X) \simeq \Omega(X)$ and $B_{G(X)} \simeq X$ the homology spectral sequence of (2.5) becomes

$$(3.2) \qquad E^2_{s,t} \cong H_s(X; \tilde{H}_t(\Sigma\Omega X)) \Rightarrow \tilde{H}_{s+t}(X).$$

Before studying the structure of this spectral sequence we must determine the fiber map $\Sigma\Omega(X) \to X$. By (2.4) the fiber map i: $(CG(X), G(X)) \rightarrowtail (D(\omega), S(\omega))$ determines the map ϕ: $\Sigma G(X) \to B_{G(X)}$, $\phi[t, g] = [(1 - t)g + te]$. Then composing with the particular homotopy equivalence $B_{G(X)} \simeq X$ constructed in [5] proves

(3.3) LEMMA. *The fiber map of the spectral sequence* (3.2) *is the map* v: $\Sigma\Omega(X) \to X$ *defined by* $v[t, \omega] = \omega(1 - t)$.

(3.4) THEOREM. *Suppose X is a k connected countable simplicial complex, where $k \geqslant 1$. Then v_*: $H_n(\Sigma\Omega X) \to H_n(X)$ is an isomorphism for $n \leqslant 2k$ and an epimorphism for $n \leqslant 2k + 1$.*

PROOF. In terms of the spectral sequence (3.2), v_* is the composite

$$\tilde{H}_n(\Sigma\Omega X) \cong E^2_{0,n} \twoheadrightarrow E^\infty_{0,n} \rightarrowtail \tilde{H}_n(X).$$

For any r, $2 \leqslant r \leqslant \infty$, we have $E^r_{s,t} = 0$ if either $0 < s \leqslant k$ or $t \leqslant k$. In particular, we always have

$$E^\infty_{1,n-1} = \cdots = E^\infty_{k,n-k} = 0.$$

On the other hand we will also have

$$E^\infty_{k+1,n-k-1} = E^\infty_{k+2,n-k-2} = \cdots = 0$$

if $n - k - 1 \leqslant k$. Thus we have proved that if $n \leqslant 2k + 1$ then $E^\infty_{0,n} \cong H_n(X)$ and so v_* is an epimorphism for this range of dimensions.

To prove that $E^2_{0,n} = E^\infty_{0,n}$ consider the differentials at $E^r_{0,n}$:

$$E^r_{r,n-r+1} \xrightarrow{d^r} E^r_{0,n} \xrightarrow{d^r} E^r_{-r,n+r-1} = 0.$$

If $n - r + 1 \leqslant k$ for all $r \geqslant k + 1$ then we have $E^2_{0,n} = E^\infty_{0,n}$. Thus we have proved that if $n \leqslant 2k$ then v_* is an isomorphism. This concludes the proof.

To relate (3.4) to the Freudenthal Suspension Theorem define $\mu = \mu_Y$: $Y \to \Omega\Sigma Y$ to be the map given by $\mu(y)(t) = [t, y]$. Also define $\lambda = \lambda_Y$: $\Sigma\Omega Y \to Y$ by $\lambda[t, \omega] = \omega(t)$. Then, by a standard fact about adjoint functors, the composite $\lambda_{\Sigma Y} \circ \Sigma\mu_Y$: $\Sigma Y \to \Sigma Y$ is the identity.

(3.5) COROLLARY. *Suppose X is a k connected countable simplicial complex, where $k \geqslant 0$. Then μ_*: $H_n(X) \to H_n(\Omega\Sigma X)$ is an isomorphism for all $n \leqslant 2k + 1$.*

PROOF. λ and v are maps $\Sigma\Omega X \to X$ that differ by a self-homeomorphism of $\Sigma\Omega X$ and therefore they have the same connectivity properties. In particular, (3.4) implies that $(\lambda_{\Sigma X})_*$: $H_n(\Sigma\Omega\Sigma X) \to H_n(\Sigma X)$ is an isomorphism for $n \leqslant 2k + 2$ and an epimorphism for $n \leqslant 2k + 3$. It follows that $(\Sigma\mu_X)_*$: $H_n(\Sigma X) \to H_n(\Sigma\Omega\Sigma X)$ is an isomorphism for $n \leqslant 2k + 2$, and this proves (3.5).

In the usual manner one can now derive the Freudenthal Suspension Theorem.

(3.6) COROLLARY. *Suppose X is an k connected countable simplicial complex, where $k \geqslant 1$. Then the suspension homomorphism Σ: $\pi_n(X) \to \pi_{n+1}(\Sigma X)$ is an isomorphism for $n \leqslant 2k$ and an epimorphism for $n \leqslant 2k + 1$.*

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