A COUNTABLE SELF-INJECTIVE RING IS QUASI-FROBENIUS

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ABSTRACT. A countable dimensional self-injective algebra is Artinian. There is an application to self-injective twisted group algebras.

It has been known for some time that a countable self-injective ring is semilocal (see for example [8]). In this paper we show that such a ring is in fact quasi-Frobenius. Special cases of this result have been proved previously, for example if the ring is also regular [3] or if it is a group algebra [8]. My thanks to Ken Louden for his help in the preparation of this paper.

Unless stated otherwise, all rings are associative with a unity. If $S$ is a subset of a ring $R$, we denote its left annihilator in $R$ by $l_R(S)$.

**Theorem 1 (Faith [1]).** A ring is quasi-Frobenius if it is right self-injective and satisfies the descending chain condition on left annihilators.

**Proposition 2.** Let $R$ be a subring of $S$. Suppose that $S_S$ is injective, $RS$ is flat and $SR$ is free. Then $RR$ is injective.

**Proof.** The proof is left to the reader.

**Theorem 3.** Every countable subring of a quasi-Frobenius ring is contained in a countable quasi-Frobenius subring. Conversely, if every countable subring of a ring is contained in a quasi-Frobenius subring, then the ring is quasi-Frobenius.

**Proof.** Suppose first that $T$ is a quasi-Frobenius ring and $A$ is a countable subring. We construct a sequence of subrings $A = R_0 \subset R_1 \subset R_2 \subset \cdots \subset T$ inductively as follows. Given $R_k$, consider all $n$-tuples $\{a_1, \ldots, a_n\}$ of elements of $R_k$ as $n$ ranges over the positive integers. If $a_n \in a_1 T + \cdots + a_{n-1} T$ choose $x_1, x_2, \ldots, x_{n-1} \in T$ so that $a_n = a_1 x_1 + \cdots + a_{n-1} x_{n-1}$. If $a_n \not\in a_1 T + \cdots + a_{n-1} T$, choose $x_n \in T$ so that $x_i a_n = 0$, $i = 1, 2, \ldots, n - 1$, and $x_n a_n \neq 0$. Now do the same for the left ideal generated by $a_1, a_2, \ldots, a_{n-1}$. Let $R_{k+1}$ be the subring of $T$ generated by $R_k$ and all the $x$'s obtained. Let $R = \bigcup_{i=1}^{\infty} R_i$. Clearly $A \subset R$ and $R$ is a countable subring, so
we need only show that $R$ is quasi-Frobenius.

As $T$ is right and left Artinian, $R$ satisfies $ACC$ and $DCC$ on right and left annihilators. If $I$ is a finitely generated right ideal of $R$, then, by construction, $r_R(I_R(I)) = I$. A dual result holds for left ideals. Thus $R$ satisfies $ACC$ on finitely generated right and left ideals and so is right and left Noetherian. As $R$ satisfies $DCC$ on right and left annihilators it is right and left Artinian. Since $R$ is right and left Artinian and satisfies the 'annihilator condition' [9, p. 276], $R$ is quasi-Frobenius.

Now suppose that every countable subring of $T$ is contained in a quasi-Frobenius subring. Then $T$ is clearly right and left Artinian. In order to prove that $T$ is right self-injective, we need only show that for all finitely generated right ideals $I$ and $J$ we have $r(I(I)) = I$ and $l(I \cap J) = l(I) + l(J)$, (see [9, p. 274]). However, if these conditions were not satisfied we could construct a countable subring $A$ such that for any subring between $A$ and $T$ these would not be satisfied, and this contradicts the hypothesis that $A$ is a subring of a quasi-Frobenius ring.

**Theorem 4.** Let $R$ be a right self-injective ring and let $\{J_i\}_{i \in \Psi}$ be a descending chain of left annihilators, well-ordered by inclusion. Then the cardinality of $\Psi$ is less than the cardinality of $R$.

**Proof of the theorem.** Suppose that the cardinality of $\Psi$ is greater than or equal to the cardinality of $R$. We may suppose that $\Psi$ is a set of ordinals. Let $\Phi$ be the set of ordinals strictly less than the cardinality of $\Psi$, thus $|\Phi| = |R|$, and we consider the descending chain of annihilators $\{J_i\}_{i \in \Phi}$. Suppose $R = \{a_j\}_{j \in \Phi}$. Suppose $J_i$ annihilates the right ideal $I_i$ on the left. Let $J = \cap_{i \in \Phi} J_i$ and let $I = \cup_{i \in \Phi} I_i$. Clearly $J$ is the left annihilator of $I$.

Consider the following proposition:

(P) For each ordinal $\alpha \in \Phi$ there is an element $b_\alpha \in I$ and an $R$-module map $\varphi_\alpha: \sum_{\rho < \alpha} b_\rho R \to R$ such that

1. If $\beta < \alpha$, then $\varphi_\alpha$ restricted to $\sum_{\rho < \beta} b_\rho R$ is $\varphi_\beta$.
2. $\varphi_\alpha(b_\alpha) \neq a_\alpha b_\alpha$.

We prove (P) by transfinite induction. For $\alpha = 1$, choose $c_1 \in J_1$ so $c_1 - a_1 \not\subset J$. Then choose $b_1 \in I$ so $(c_1 - a_1)b_1 \neq 0$. Let $\varphi_1$ be left multiplication by $c_1$.

Now suppose we have proved (P) for all ordinals less than $\delta$. We have a right module homomorphism

$$\varphi'_\delta: \sum_{\rho < \delta} b_\rho R \to R,$$

simply given by the union of the $\varphi_\rho$, $\rho < \delta$. As $R$ is right self-injective, $\varphi'_\delta$ is given by left multiplication, say by $d_\delta$. Let $x$ be an ordinal large enough so $\{b_j\}_{j < x} \subset I_x$. Choose $c_\delta \in J_x$ so $c_\delta + d_\delta - a_\delta \not\subset J$, and then choose $b_\delta$ so that $(c_\delta + d_\delta - a_\delta)b_\delta \neq 0$. Define $\varphi_\delta$ to be left multiplication by $c_\delta + d_\delta$. Thus (P) is proved by transfinite induction.

Let $\varphi: \sum_{\rho \in \Phi} b_\rho R \to R$ be the right $R$-module map defined by the union of
the \( \varphi_p \). Then for all \( \alpha \in \Phi \), \( \varphi \) restricted to \( \sum_{p \leq \alpha} b_p R \) is simply \( \varphi_\alpha \). Therefore \( \varphi(b_\alpha) = \varphi_\alpha(b_\alpha) \neq a_\alpha b_\alpha \); hence, \( \varphi \) is not given by left multiplication, contradicting the hypothesis that \( R \) is right self-injective. This completes the proof of the theorem.

**Proposition 5.** Let \( A \) be an infinite set. Then there is a totally ordered (by inclusion) subset of the power set of cardinality \( 2^{|A|} \).

The above proposition allows us to construct the following example. Let \( F \) be a countable field and let \( A \) be an infinite set of ordinals less than a given cardinality. Let \( R_A = \bigoplus_{\alpha \in A} F \) be the direct product of \( A \) copies of \( F \). Then \( R \) is self-injective and \( |R_A| = 2^{|A|} \). Also, \( R_A \) has a well-ordered descending chain of annihilators of cardinality \( |A| \) and a totally ordered descending chain of annihilators of cardinality \( 2^{|A|} \). This example shows that 'well ordered' cannot be replaced by 'totally ordered' in the theorem.

**Theorem 6.** Let \( T \) be a right self-injective ring such that every countable subring is contained in a countable subring \( R \), where \( T \) is free as a right \( R \)-module and flat as a left \( R \)-module. Then \( T \) is quasi-Frobenius.

**Proof.** By Proposition 2 and Theorem 3, it is enough to show that a countable right self-injective ring is quasi-Frobenius.

**Corollary 7.** A countable dimensional self-injective algebra over a field is quasi-Frobenius.

**Corollary 8** (Renault). A group algebra is self-injective only if the group is finite.

**Proof.** A self-injective group algebra is quasi-Frobenius, hence Artinian, so the group is finite.

**Corollary 9.** A ring is quasi-Frobenius if and only if every countable subring is contained in a countable self-injective subring.

**Proof.** This is an easy consequence of Theorems 3 and 4.

If we look at rings without a unity, then most of the above theorems fail to hold. Let \( S \) denote the semigroup \( \{ e_1, e_2, \ldots : e_i e_j = e_i \} \). If \( F \) is any field, then the semigroup ring \( FS \) is left but not right self-injective and is neither right nor left Artinian.

Recall that a twisted group algebra \( F^t G \) is defined by a 2-cocycle \( t: G \times G \to F - \{0\} \), where \( G \) is a group and \( F \) is a field, and where we define \( \tilde{g} \cdot \tilde{h} = t(g, h) \tilde{g} \tilde{h} \). Define the cocycle subfield of \( F \) to be the subfield generated by the image of \( t \). Passman has constructed an example of an infinite group such that for certain fields the twisted group algebra is a field. In the same paper [6], Passman proved that if \( F \) is algebraically closed and uncountable and \( F^t G \) is Artinian, then \( G \) is finite. We use his idea in the following theorem.

**Theorem 10.** Suppose that \( F^t G \) is a self-injective twisted group algebra such
that $F$ is a proper extension of the algebraic closure of the cocycle subfield. Then $G$ is finite.

Proof. If $G$ is not finite, then we may assume that it is countably infinite [7], hence $F'G$ is quasi-Frobenius. Let $\Delta(G)$ denote the set of elements in $G$ with finitely many conjugates. Then $F'\Delta(G)$ is self-injective, so $\Delta(G)$ is finite [7]. Now using an argument similar to Passman's [6, p. 648] we may assume that $F'G$ is Artinian and $\Delta(G) = \langle 1 \rangle$. Let $K$ denote the cocycle subfield of $F$ and let $L$ denote the algebraic closure of $K$ in $F$. Clearly

$$F'G \cong F \otimes_L L'G,$$

and as $F$ is not algebraic over $L$, $L'G$ must be an algebraic $L$-algebraic [4]. By a Theorem of Passman, $L'G$ is a semiprime [5, p. 424], so $L'G$ is a semiprime Artinian algebraic algebra over an algebraically closed field. Therefore, $G$ is finite.

References

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