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CONTINUOUS ERGODIC MEASURES ON R^{∞} HAVE DISJOINT POWERS

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ABSTRACT. If μ is an ergodic probability measure on an infinite dimensional linear measure space and if there exists an infinite sequence of measurable linear functionals on this space such that all nontrivial linear combinations have continuous distribution under μ, then the convolution powers of μ all live on disjoint sets.

1. **Introduction**. Suppose μ_1 and μ_2 are probability measures on R^{∞}. We say μ_1 and μ_2 are disjoint if there exist two disjoint Borel subsets A_1, A_2 of R^{∞} with $\mu_i(A_i) = 1$. If μ is a probability measure on R^{∞}, we let μ^n stand for the nth convolution power of μ and for $x \in R$ we let $\tau_x \circ \mu$ stand for the measure μ convoluted with δ_x, unit mass at x. A natural question (from the stochastic processes point of view) is to ask if the measures μ^n are strongly disjoint (i.e. if for any sequence $x_n \in R^{\infty}$ the measures $\tau_{x_n} \circ \mu^n$ are pairwise disjoint).

This question obviously has a negative response in general; however it turns out that the question can be answered affirmatively for a large class of measures μ. To be specific, we shall prove that μ has strongly disjoint powers if it satisfies the conditions

(A) that any finite sum of the form $\Sigma_1^n a_k x_k$ with a_k real and not all zero has a continuous distribution, and

(B) that μ is ergodic.

(Here x_k stands for the kth coordinate of an element x in R^{∞} and we follow Brown and Moran [3] in defining a measure μ to be ergodic if there exists a countable subgroup D of R^{∞} such that μ is D ergodic, i.e. such that any D invariant set has μ measure 1 or 0.)

Some very special cases of our result are already known. For instance, if μ is a product measure $\otimes_1^{\infty}\mu_i$ where each μ_i is a continuous probability measure on R, then an application of Kakutani's equivalence-singularity dichotomy [9, p. 81] will show that the measures μ^n are strongly disjoint. In particular this works for any Gaussian measure μ on R^{∞} which satisfies condition (A), since we can map μ into a product measure by a 1-1 linear transformation of R^{∞} into itself (the Gram-Schmidt orthonormalization procedure will effect this, for instance).

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Our method of proof deserves some comment in that it makes essential use of Banach algebra techniques, in particular the use of the maximal ideal space of a Banach algebra and its Gelfand topology (see [11]), the notions of generalized characters [10] and Shilov boundary [11]. It may be useful for probabilists to achieve some acquaintance with these techniques in the context of solving a problem in stochastic processes which seems to lie outside the scope of the methods usually employed in probability theory.

Before going on with the proof of our theorem we establish first some conventions about notation and terminology. If μ_1, μ_2 are probability measures on any measurable space we shall write $\mu_1 \gg \mu_2$ (or $\mu_1 \approx \mu_2$) if null sets for μ_1 are also null sets for μ_2 (if μ_1 and μ_2 have the same null sets).

We shall replace R^∞ by a measurable abelian group, i.e. by any abelian group S which is endowed with a σ-field $\mathcal{Q}_S$ such that the map $(x, y) \to x + y$ is measurable with respect to the product σ-field $\mathcal{Q}_S \times \mathcal{Q}_S$. We shall assume henceforth that μ is a *separable* measure on $\mathcal{Q}_S$ (that a countable sequence of sets in $\mathcal{Q}_S$ generates $\mathcal{Q}_S$ up to μ null sets).

We now present some of the notions we are borrowing from the theory of Banach algebras. (See [11] for general background on this subject.) We call a collection M of bounded complex valued measures on S an L-algebra if M is a Banach algebra where multiplication is convolution of measures and the norm is the total variation norm, and if it is true that for any two bounded complex valued measures ν_1, ν_2 on S with ν_2 in M we can conclude that ν_1 is also in M if $\nu_2 \gg \nu_1$ holds. It is well known that the maximal ideal space $\Delta(M)$ (the set of all bounded complex valued multiplicative linear functionals on any Banach algebra M) is locally compact in the Gelfand topology (and compact if M has unit). When M is an L-algebra then the work of Šreĭder [10] shows that any $\phi \in \Delta(M)$ is of the form

$$\phi(\nu) = \int_S \chi_\nu d_\nu$$

where $(\chi_\nu)_{\nu \in M}$ is a *generalized character* of M, i.e. a member χ of the product space $\Pi_{\nu \in M} L^\infty(\nu)$ satisfying the following conditions:

(C_1) If $\nu_1 \ll \nu_2$ then $\chi_{\nu_1} = \chi_{\nu_2}$($\nu_1$ a.e.).

(C_2) $\chi_{\nu_1 * \nu_2}(x + y) = \chi_{\nu_1}(x)\chi_{\nu_2}(y)$ ($\nu_1 \times \nu_2$ a.e.), for ν_1, ν_2 in M.

(C_3) $1 \geqslant \sup_{\nu \in M} \|\chi_\nu\|_\infty > 0$, where $\|\chi_\nu\|_\infty = \text{l.u.b}_K\{K > |\chi_\nu(x)|(\nu \text{ a.e.})\}$.

If μ is any finite nonnegative measure on S we shall denote by $M(\mu)$, the smallest L-algebra of measures on S which contains μ and all discrete measures in S. We shall denote by $M_0(\mu)$ the smallest L-algebra of measures on S which contains μ. It is obvious that $\nu \in M_0(\mu)$ if and only if $\nu \ll \Sigma_1^\infty (2\mu(S))^{-n}\mu^n$.

We now present two lemmas due to Brown and Moran which are crucial to our main result.

LEMMA 1.1. *Let μ be an ergodic measure on S. Then $|\chi_\mu|$ is a constant function (μ a.e.) for all χ in $\Delta(M(\mu))$.*

The proof of the above lemma can be found in [3, p. 309]; the proof in [3] assumes the extra condition that $\mu \approx \mu * \delta_x$ for each $x \in D$. However if $\{d_n;\ n = 1, 2, \ldots\}$ is an enumeration of D then the measure

$$\mu' = \sum_{n=1}^{\infty} 2^{-n}\mu * \delta_{d_n}$$

satisfies this extra condition, is D ergodic, and has $M(\mu) = M(\mu')$ as well as $\chi_\mu = \chi_{\mu'}$ (μ a.e.) for all χ. We thank Gavin Brown for this remark.

The next lemma's proof can be pieced together from [3, p. 309] and [2, p. 303]. We supply the proof for the reader's convenience.

LEMMA 1.2. *Let μ be an ergodic measure on S. Let H stand for the set of elements χ in $\Delta(M(\mu))$ such that $|\chi_\mu| = 1$ (μ a.e.). Then either μ has strongly disjoint powers or H is open in the Gelfand topology.*

PROOF. Assume that μ does not have strongly disjoint powers, i.e. that there exists distinct positive integers p and q and a nonzero measure ν such that $\nu \ll \mu^p$ and $\nu \ll \tau_s \circ \mu^q$ for $s \in S$. Let $\chi \in \Delta(M(\mu))$. By Lemma 1.1 we know there exists a real number c in $[0, 1]$ such that $|\chi_\mu(x)| = c$ (μ a.e.). Using (C_1) and (C_2) we get that

$$\begin{aligned} c^p &= |\chi_{\mu^p}(x)| = |\chi_\nu(x)| = |\chi_{\tau_s \circ \mu^q}(x)| \\ &= \left(|\chi_{\delta(s)}(x)|\right)\left(|\chi_{\mu^q}(x)|\right) = |\chi_{\mu^q}(x)| = c^q \qquad (\nu \text{ a.e.}) \end{aligned}$$

since $|\chi_{\delta(s)}| = 1$ (as follows from $\delta_{(s)} * \delta_{(-s)} = \delta_0$). We conclude that c is either 0 or 1. It now readily follows that H is an open subset of $\Delta(M(\mu))$ (in the Gelfand topology). Q.E.D.

In anticipation of future use of Lemma 1.2 we now note that the set of generalized characters of any L-algebra form a semigroup under pointwise multiplication, i.e. that multiplication is separately continuous in the Gelfand topology. It is clear that H is a subgroup of $\Delta(M(\mu))$ and in fact Brown and Moran [3] show that H is a locally compact abelian group (i.e. that multiplication is jointly continuous on H). We do not need this fact but borrow their reasoning in Theorem 2.1 to prove something very similar.

2. **Measures with disjoint powers.** In this section we present our principal results. From now on, linear space structure will be essential so we shall assume that the measurable group S is also a *measurable linear space* over the reals, i.e. the map $(a, x) \to ax$ is measurable from $R \times S$ into S with respect to the σ-field which is the product of Borel sets in R with $\mathcal{Q}_S$. The following lemma is proved in [5].

LEMMA 2.1. *Let G be a locally compact abelian group which is also a real vector space. Assume also that for each real a and each $\hat{g}$ in the character group $\hat{G}$ of G the map $g \to \hat{g}(ag)$ is continuous. Assume also that the map $a \to \hat{g}(ag)$ is a character of R. Then G is a finite dimensional vector space.*

The road to our theorem is almost clear. We need only a generalized version of condition (A). We say that a probability measure on the measurable linear space S satisfies condition (A′) if there exists an infinite sequence f_n of μ completion measurable real linear functionals on S such that for any real sequence a_k, not all 0, the sum $\Sigma_1^n a_k f_k$ has a continuous distribution.

THEOREM 2.1. *Let μ be a separable probability measure on the measurable linear space S. Assume μ is ergodic and satisfies condition* (A′). *Then μ has strongly disjoint powers.*

PROOF. Assume μ does not have strongly disjoint powers. Let H be as defined in Lemma 1.2. Let G be the group of elements χ of $\Delta(M_0(\mu))$ such that $|\chi_\mu| = 1$ (μ a.e.). Let θ be the map from $\Delta(M(\mu))$ into $\Delta(M_0(\mu))$ defined by restricting multiplicative linear functions on $M(\mu)$ to $M_0(\mu)$. It is obvious that θ is continuous. Furthermore any element of G is in the Shilov boundary of $\Delta(M_0(\mu))$ by [1], hence every element of G can be extended to be a multiplicative linear functional on $M(\mu)$, as is well known (see e.g. [11]). It follows that $\theta(H) = G$.

For $x, y \in \Delta(M(\mu))$ we define the equivalence relation $\sim$ by saying $x \sim y$ iff $\theta(x) = \theta(y)$. We let Δ stand for the set of equivalence classes of $\Delta(M(\mu))$ endowed with the largest topology which makes the map $Q(x) = [x]$ continuous, where $[x]$ stands for the equivalence class containing x.

We can define $\hat{\theta}$: $\Delta \to \Delta(M_0(\mu))$ by the equation $\hat{\theta}Q = \theta$. It follows that $\hat{\theta}$ is a 1-1 continuous map from Δ into $\Delta(M_0(\mu))$ where $\Delta(M_0(\mu))$ has the Gelfand topology. We have $\hat{\theta}(Q(H)) = G$ and $Q(H)$ is open in Δ (since $H = Q^{-1}(Q(H))$ is open by Lemma 1.2). Since $\hat{\theta}$ is a 1-1 continuous map defined on a compact set it follows that $\hat{\theta}(Q(H))$ is open in $\hat{\theta}(Q(\Delta(M(\mu))))$ $= \theta(\Delta(M(\mu)))$. It follows that $G = \hat{\theta}(Q(H))$ is locally compact since it is an open subset of the compact set $\theta(\Delta(M(\mu)))$. Also multiplication is separately continuous in $\Delta(M_0(\mu))$ hence G is a locally compact abelian group by applying Ellis's theorem [4].

We can now get a contradiction to our assumption that μ did not have strongly disjoint powers. Let $f_1, \ldots, f_n, \ldots$ be an infinite sequence of μ completion measurable linear functionals on S as in condition (A′). For $h = (h_1, \ldots, h_n, \ldots)$ in R_∞ let $J(h)$ denote that element χ in $\Delta(M_0(\mu))$ such that $\chi_\nu(x) = \exp(i\Sigma h_k f_k(x))$ for all $x \in S$ and all $\nu \in M_0(\mu)$. By condition (A′) we conclude that J is a 1-1 map of R_∞ into G. Consider now G_0, the closure of $J(R_\infty)$ in G. If $\lambda = \Sigma_{n=1}^\infty 2^{-n}\mu^n$ then $L_1(\lambda)$ is separable, and the dual of $L_1(\lambda)$ has a countable neighbourhood basis at every point. Since $\Delta(M_0(\mu))$ sits in the dual of $L_1(\lambda)$ we conclude that G_0 is the sequential closure of $J(R_\infty)$ in G. Thus for $g \in G_0$ there exists a sequence $g_1, \ldots, g_n, \ldots$ in $J(R_\infty)$ such that $g_n \to g$ in the Gelfand topology.

For $a \in R$ and $g = J(h)$ in $J(R_\infty)$ define $a \cdot g = J(ah)$. If $g \in f_0$ and $g_n \to g$ as above define $a \cdot g = \lim_{n\to\infty} a \cdot g_n$. We now see that G_0 has been made into a real vector space (since $a \cdot g \neq 0$ if $a \neq 0$ and $g \neq 0$ by condition (A′)).

Now G_0 is a closed subgroup of G so G_0 is also a locally compact abelian group. For each $\hat{g}$ in $\hat{G}_0$ and each $a \in R$ the map $g \to \hat{y}(a \cdot g)$ is clearly continuous. Also the map $a \to \hat{y}(a \cdot g)$ is Borel measurable from R into the complex numbers of modulus no greater than 1 since it is the pointwise limit of Borel measurable functions, hence it is a continuous character of R by [7, Theorem 9.3.1]. By Lemma 2.1 we conclude that G_0 is finite dimensional, which is a contradiction, since G_0 contains $J(R_\infty)$ as a vector subspace. Q.E.D.

Theorem 2.1 applies, of course, in the special case $S = R^\infty$ as claimed in the introduction. In the following example we let D stand for the set of x in R^∞ with rational coordinates x_k such that $x_k = 0$ for k sufficiently large.

EXAMPLE 2.1. Let Y_k be a sequence of independent random variables with absolutely continuous distributions. Let $(c_{nk};\ n \geqslant 1, k \geqslant 1)$ be an array of real numbers such that the infinite sums $X_n = \Sigma_k c_{nk} Y_k$ converge a.s. for all $n \geqslant 1$. Suppose that the vectors $C_n = (c_{nk};\ k \geqslant 1)$ are not contained in a finite dimensional linear subspace of R^∞. Letting μ stand for the probability measure induced on R^∞ by the stochastic process $(X_n;\ n \geqslant 1)$, we claim that μ is ergodic and satisfies condition (A).

We first show that μ is ergodic. Let $U = \{y;\ y \in R^\infty, \Sigma_k c_{nk} y_k$ is convergent for all $n\}$. For $y \in U$ define $T(y)$ to be that vector in R^∞ whose nth coordinate is $\Sigma_k c_{nk} y_k$. Letting ν stand for the measure induced on R^∞ by the process $(Y_1, Y_2, \ldots, Y_n, \ldots)$ we note that $\nu(U) = 1$ and $\mu = \nu \circ T^{-1}$. Let $D' = T(D)$.

We claim that μ assigns measure 0 or 1 to any D' invariant set. In fact if A in $\mathfrak{B}^\infty$ is D' invariant then $T^{-1}(A)$ is D invariant, whence by the Kolmogorov 0-1 law it follows that $\mu(A) = \nu(T^{-1}(A)) = 0$ or 1. It follows that μ is ergodic.

To check that μ satisfies condition (A) is trivial since if C_{n_j} is a linearly independent infinite subsequence of the vectors C_n then for any finite nonzero sequence $a_1, \ldots, a_N$, the sum $\Sigma_1^N a_j X_{n_j}$ has a continuous distribution (since it is the sum of the independent random variables $(\Sigma_{j=1}^N a_j c_{n_j}) Y_k$, some of which must have continuous distribution).

In conclusion we now make some comments on the necessity of ergodicity or condition (A′) in Theorem 2.1. We shall let $\mathfrak{B}^\infty$ stand for the Borel subsets of R^∞.

To show that ergodicity is necessary we consider the following example, which is in marked contrast to the case of μ Gaussian (see Example 2.4).

EXAMPLE 2.2. Let Y_k be a sequence of independent identically distributed Gaussian random variables of mean 0 and variance 1. Let Z be independent of the sequence Y_k and suppose that Z is distributed like $|Y_k|^{-1}$.

Consider the sequence $X_k = (Z) Y_k$ and let μ be the measure that the X_k process induces on R^∞. For a_k real, any sum $\Sigma_1^n a_k X_k$ is Cauchy by [6, p. 176] so $(\phi_a \circ \mu) * (\phi_b \circ \mu) = \phi_{a+b} \circ \mu$ for all $a, b > 0$ (where $\phi_a \circ \mu$ denotes the measure defined by $\phi_a \circ \mu(B) = \mu(a^{-1}B)$ for $B \in \mathfrak{B}^\infty$). In particular it follows that μ^n is the measure induced by the process $(nX_k;\ k \geq 1)$ for any positive integer n. Now the random variable Z has positive density on $(0, \infty)$ hence the distribution of Z and nZ are equivalent measures on $(0, \infty)$ for any $n \geq 1$. This implies that the processes $(X_k;\ k \geq 1)$ and $(nX_k;\ k \geq 1)$ induce equivalent measures on R^{∞} so μ^n are all equivalent. It is obvious that μ satisfies condition (A) so we conclude that μ is not ergodic.

We now give an example to show the necessity of condition (A′).

EXAMPLE 2.3. Let μ on R^{∞} be a product measure $\bigotimes_1^{\infty}\mu_i$ where μ_i assigns mass ρ_i to 0 and mass $(1 - \rho_i)$ to 1. If $\Sigma(1 - \rho_i) < \infty$ then both μ and μ^2 assign positive mass to $(0, 0, \ldots, 0, \ldots)$, hence μ does not have strongly disjoint powers. Clearly μ is ergodic and condition (A′) is not satisfied.

The following example considers the special case of Gaussian measures.

EXAMPLE 2.4. We shall say that μ is an infinite dimensional symmetric Gaussian measure on R^{∞} if all finite sums $\Sigma_1^n a_k x_k$ have a symmetric Gaussian distribution and if condition (A) is satisfied for some sequence f_n as before. Applying the Gram-Schmidt orthonormalization procedure to f_n we get an infinite sequence f_n' of independent $N(0, 1)$ random variables defined on (R^{∞}, μ). Note now that on (R^{∞}, μ^k) the f_n' are a sequence of independent $N(0, k)$ random variables hence it is clear that μ has disjoint powers. In summary, for Gaussian measures condition (A) is sufficient to imply disjoint powers. (By a similar argument condition (A′) is also sufficient.)

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