A NOTE ON TWO CONGRUENCES ON A GROUPOID

K. NIRMALA KUMARI AMMA

Abstract. Let S be a groupoid and \( \theta_p, \theta_m \) the congruences on S defined as follows: \( x \theta_p y (x \theta_m y) \) iff every prime (minimal prime) ideal of S containing \( x \) contains \( y \) and vice versa. It is proved that \( \theta_p \) is the smallest congruence on S for which the quotient is a semilattice. It is also shown that \( S/\theta_m \) is a disjunction semilattice if S has 0 and is a Boolean algebra if S is intraregular and closed for pseudocomplements. Some connections between the ideals of S and those of the quotients are established. Congruences similar to \( \theta_p \) and \( \theta_m \) are defined on a lattice using lattice-ideals; quotients under these are distributive lattices.

Introduction. This paper consists of four sections. §1 is devoted to a summary of some results on ideals and pseudocomplements which are used in subsequent sections. In §2 we define two congruences on a groupoid S in terms of prime ideals. The quotients under these are semilattices. Relations between the ideals of S and those of the quotients are investigated. Theorems 3 and 6 are the main theorems in this direction. §3 deals with the additional properties of the congruences. Two congruences on a lattice are given in §4. Some connected results for commutative semigroups and lattices have been proved by Kist [3], Varlet [5] and Venkatanarasimhan [6]. For standard concepts and notations used in this paper, the reader may refer to Birkhoff [1] and Clifford and Preston [2].

1. Preliminaries. By an ideal we mean an ideal in the sense of [2] and by a lattice-ideal we mean an ideal in the sense of [1]. Let S be a groupoid. A nonempty subset \( A \) of S is called a filter if \( a, b \in A \Leftrightarrow ab \in A \). The smallest filter containing an element \( a \) is called the principal filter generated by \( a \) and is denoted by \( K(a) \). The principal ideal generated by \( a \) is denoted by \( J(a) \). An ideal \( [A] \) of S is said to be prime if \( ab \in A \Rightarrow a \in A \) or \( b \in A \). S is said to be intraregular if \( J(ab) = J(a) \cap J(b) \) for any two elements \( a, b \) of S. If S has 0 and \( a \in S \), by the pseudocomplement of \( a \) we mean an element \( a^* \) of S such that \( aa^* = 0 = a^*a \) and \( ab = 0 \Rightarrow ba^* = b = a^*b \). We shall denote lattice-join and lattice-meet by the symbols \( \vee \) and \( \wedge \) respectively.

We need the following lemmas in the sequel.

Lemma I [4]. A nonempty proper subset \( A \) of a groupoid S is a prime (minimal...
prime) ideal of $S$ if and only if $S - A$ is a filter (maximal filter) of $S$.

**Lemma II** [4]. Any prime ideal (proper filter) of a groupoid with 0 contains (is contained in) a minimal prime ideal (maximal filter).

**Lemma III** [4]. A groupoid $S$ is intraregular if and only if every ideal of $S$ is an intersection of prime ideals.

**Lemma IV** [4]. In an intraregular groupoid $S$ with 0, a prime ideal is minimal prime if and only if it contains precisely one $J(x), J(x)^*$ for every $x \in S$. In an intraregular groupoid $S$ closed for pseudocomplements, a prime ideal is minimal prime if and only if it contains precisely one of $x, x^*$ for every $x \in S$.

**Lemma V** [4]. In an intraregular groupoid with 0, the pseudocomplement of an ideal is the intersection of all the minimal prime ideals not containing it.

**Lemma VI** [7]. A semilattice $S$ with 0 is a disjunction semilattice if and only if distinct principal ideals of $S$ have distinct pseudocomplements.

**Lemma VII** [7]. A disjunction semilattice closed for pseudocomplements is a Boolean algebra.

**Lemma VIII** (cf. [2, p. 12, Example 1]). Let $f$ be a homomorphism of a groupoid $S$ onto a groupoid $T$. Then:

(i) If $A$ is an ideal of $S$, $f(A)$ is an ideal of $T$. If $B$ is an ideal of $T$, $f^{-1}(B)$ is an ideal of $S$.

(ii) If $A$ is a prime ideal (filter) of $T$, $f^{-1}(A)$ is a prime ideal (filter) of $S$.

(iii) $f(J(x)) = J(f(x))$ for all $x \in S$.

Proofs are not given for the results on filters since they may be inferred from the ones for prime ideals using Lemma I.

2. Two congruences on a groupoid. Throughout this section, $S$ will denote a groupoid.

For $x, y \in S$ define $x \theta_p y (x \theta_m y)$ to mean that every prime (minimal prime) ideal of $S$ containing $x$ contains $y$ and vice versa. It is easily seen that $\theta_p$ and $\theta_m$ are congruences on $S$ and that $S/\theta_p$ and $S/\theta_m$ are semilattices. We shall denote the natural homomorphism of $S$ onto $S/\theta_p (S/\theta_m)$ by $\theta_p^*(\theta_m^*)$.

**Lemma 1.** Let $A$ be a prime ideal or a filter (minimal prime ideal or a maximal filter) of $S$. Then $\theta_p^* \theta_m^*(A) = A$ ($\theta_m^* \theta_p^*(A) = A$).

**Proof.** Clearly the prime (minimal prime) ideals of $S$ are unions of $\theta_p$ ($\theta_m$)-classes. Hence the lemma follows.

**Theorem 1.** If $A$ is any prime ideal (filter) of $S$, then $\theta_p^*(A)$ is a prime ideal (filter) of $S/\theta_p$. If $A$ is a minimal prime ideal (maximal filter) of $S$, then $\theta_p^*(A)$ is a minimal prime ideal (maximal filter) of $S/\theta_p$ and $\theta_m^*(A)$ is a minimal prime ideal (maximal filter) of $S/\theta_m$.

**Proof.** Let $A$ be a prime ideal of $S$. Clearly, $x, y \in S$, $\theta_p^*(x)\theta_p^*(y) \in \theta_p^*(A)$.
If \( A \) is a minimal prime ideal and \( B \) is any prime ideal of \( S/\theta_p \) such that \( B \subseteq \theta_p^*(A) \), then \( \theta_p^{*-1}(B) \subseteq A \) by Lemma 1. Also \( \theta_p^{*-1}(B) \) is prime by Lemma VIII. Hence \( \theta_p^{*-1}(B) = A \) and so \( \theta_p^*(A) = B \). Thus \( \theta_p^*(A) \) is minimal prime. The last part is proved on similar lines.

**Theorem 2.** If \( A \) is a minimal prime ideal (maximal filter) of \( S/\theta_p \), then \( \theta_p^{*-1}(A) \) is a minimal prime ideal (maximal filter) of \( S \). If \( S \) has 0 and \( A \) is a minimal prime ideal (maximal filter) of \( S/\theta_m \), then \( \theta_m^{*-1}(A) \) is a minimal prime ideal (maximal filter) of \( S \).

**Proof.** Let \( A \) be a minimal prime ideal of \( S/\theta_p \). By Lemma VIII, \( \theta_p^{*-1}(A) \) is a prime ideal of \( S \). Let \( B \) be any prime ideal of \( S \) such that \( \theta_p^{*-1}(A) \supseteq B \). Then \( A \supseteq \theta_p^*(B) \). By Theorem 1 and Lemma 1, it follows that \( \theta_p^{*-1}(A) = B \). Hence \( \theta_p^{*-1}(A) \) is a minimal prime ideal of \( S \).

Suppose \( S \) has 0 and \( A \) is a minimal prime ideal of \( S/\theta_m \). By Lemma VIII, \( \theta_m^{*-1}(A) \) is prime. By Lemma II, \( \theta_m^{*-1}(A) \supseteq B \) for some minimal prime ideal \( B \) of \( S \). As in the first part, \( \theta_m^{*-1}(A) = B \). Thus, \( \theta_m^{*-1}(A) \) is minimal prime.

As a consequence of Theorems 1 and 2 and Lemma VIII, we have the following.

**Theorem 3.** There is a bijection between the set of prime ideals (filters) of \( S \) and the set of prime ideals (filters) of \( S/\theta_p \). Under this bijection, the minimal prime ideals (maximal filters) of \( S \) correspond to the minimal prime ideals (maximal filters) of \( S/\theta_p \). If \( S \) has 0, there is a bijection between the set of minimal prime ideals (maximal filters) of \( S \) and the set of minimal prime ideals (maximal filters) of \( S/\theta_m \).

**Theorem 4.** If \( \{A_i; i \in I\} \) is a family of prime ideals or filters (minimal prime ideals or maximal filters) of \( S \), then

\[
\theta_p^* \left( \bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} \theta_p^*(A_i) \neq \theta_m^* \left( \bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} \theta_m^*(A_i).
\]

**Proof.** Let \( x \in \bigcap_{i \in I} \theta_p^*(A_i) \). Then for each \( i \in I \), there exists \( x_i \in A_i \) such that \( x = \theta_p^*(x_i) \). It follows that \( x \in \theta_p^*(\bigcap_{i \in I} A_i) \). Thus \( \bigcap_{i \in I} \theta_p^*(A_i) \subseteq \theta_p^*(\bigcap_{i \in I} A_i) \). The reverse inclusion is obvious.

The second part is proved on similar lines.

As a consequence of Lemmas 1 and III and Theorem 4, we have the following.

**Corollary 1.** If \( S \) is intraregular and \( A \) is an ideal of \( S \), then \( \theta_p^{*-1}(A) = A \).

**Corollary 2.** If \( S \) is intraregular and \( \{A_i; i \in I\} \) is a family of ideals of \( S \), then \( \theta_p^*(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} \theta_p^*(A_i) \).

The following theorem is easily proved.
Theorem 5. For each \( x \in S \), \( \theta_p^*(K(x)) = K(\theta_p^*(x)) \). If \( \{ A_i ; i \in I \} \) is a family of filters of \( S \), \( \theta_p^*(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} \theta_p^*(A_i) \). If \( \{ B_i ; i \in I \} \) is a family of filters of \( S/\theta_p \), \( \theta_p^{* -1}(\bigvee_{i \in I} B_i) = \bigvee_{i \in I} \theta_p^{* -1}(B_i) \).

As an immediate consequence of Theorems 3, 4 and 5, Corollaries 1 and 2 and Lemma VIII, we have the following.

Theorem 6. If \( S \) has 1, \( \theta_p^* \) induces an isomorphism between the lattice of filters of \( S \) and the lattice of filters of \( S/\theta_p \) which preserves unrestricted joins, unrestricted meets, the property of being a principal filter and the property of being a maximal filter in both directions. If \( S \) is an intraregular groupoid with 0, \( \theta_p^* \) induces an isomorphism between the lattice of ideals of \( S \) and the lattice of ideals of \( S/\theta_p \) which preserves unrestricted joins, unrestricted meets, the property of being a principal ideal, the property of being a prime ideal and the property of being a minimal prime ideal in both directions.

Theorem 1. If \( S \) is an intraregular groupoid with 0 and \( A \) is any ideal of \( S \), then \( \theta_m^*(A^*) = \theta_m^*(A)^* \).

Proof. Let \( \{ A_i ; i \in I \} \) be the family of minimal prime ideals of \( S \) not containing \( A \). Then the ideals \( \theta_m^*(A_i) \) are precisely the minimal prime ideals of \( S/\theta_m \) not containing \( \theta_m^*(A) \). By Lemma V and Theorem 4 it follows that \( \theta_m^*(A^*) = \theta_m^*(A)^* \).

3. Additional properties of \( \theta_p \) and \( \theta_m \). As in §2, \( S \) will denote a groupoid throughout this section.

Theorem 8. Let \( \theta \) be any congruence on \( S \). Then \( S/\theta \) is a semilattice if and only if \( \theta_p \subseteq \theta \).

Proof. The 'If part' is obvious. Suppose \( S/\theta \) is a semilattice and let \( x \theta_p y \). Let \( \theta^* \) denote the natural homomorphism of \( S \) onto \( S/\theta \). It is easily seen that \( J(\theta^*(x)) = J(\theta^*(y)) \). Hence \( \theta^*(x) = \theta^*(y) \). Thus \( \theta_p \subseteq \theta \).

Theorem 9. For any two distinct elements of \( S/\theta_m \), there exists a minimal prime ideal of \( S/\theta_m \) containing exactly one of them.

Proof. Let \( \theta_m^*(x), \theta_m^*(y) \) be any two distinct elements of \( S/\theta_m \) so that \( x, y \in S \). Then there exists a minimal prime ideal \( A \) of \( S \) containing exactly one of \( x, y \), say \( x \). Clearly, \( \theta_m^*(A) \) is a minimal prime ideal of \( S/\theta_m \) such that \( \theta_m^*(x) \in \theta_m^*(A) \) and \( \theta_m^*(y) \notin \theta_m^*(A) \).

As a consequence of Theorem 9 and Lemmas IV and VI, we have the following.

Theorem 10. If \( S \) is an intraregular groupoid with 0, then \( S/\theta_m \) is a disjunction semilattice.

Theorem 11. Let \( S \) be an intraregular groupoid closed for pseudocomplements. Then \( x \theta_m y \leftrightarrow x^* = y^* \) and \( S/\theta_m \) is a Boolean algebra.

Proof. By Lemma IV it follows that \( x \theta_m y \leftrightarrow x^* = y^* \). It is easily seen that
$\theta^*_m(x^*)$ is the pseudocomplement of $\theta^*_m(x)$ for each $x \in S$. By Theorem 10 and Lemma VII it follows that $S/\theta^*_m$ is a Boolean algebra.

4. **Two congruences on a lattice.** Throughout this section $L$ will denote a lattice.

For $x, y \in L$ define $x \phi_p y$ ($x \phi_m y$) to mean that every prime lattice-ideal (minimal prime lattice-ideal) containing $x$ contains $y$ and vice versa. Then it is easily seen that $\phi_p$ and $\phi_m$ are congruences on $L$ and that $L/\phi_p$ and $L/\phi_m$ are distributive lattices.

Analyses of many of the results in §§2 and 3 hold for $\phi_p$ and $\phi_m$. For example, we have the following.

**Theorem 12.** Let $\theta$ be any congruence on $L$. Then $L/\theta$ is a distributive lattice if and only if $\phi_p \subseteq \theta$.

**Acknowledgement.** The author is indebted to Dr. P. V. Venkatanarasimhan for his guidance and help in preparing this note. She is thankful to the referee for some suggestions.

**References**


**Department of Mathematics, University of Kerala, Kariavattom, Trivandrum, India**