RELATIVE S-INVARIANTS

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ABSTRACT. Warfield has defined the concept of a $T^*$-module over a discrete valuation ring and has proved a classification theorem for these modules. In this paper, the invariant $S$ defined by the author is extended. This allows a generalization of the classification theorem of Warfield.

1. Preliminaries. $R$ will be a discrete valuation ring throughout, and $p$ will represent a generator of its maximal ideal. The word “module” will mean $R$-module. Standard terminology of abelian groups will be used (see Fuchs [1], especially §79). The exception to this is that we will define the height of 0 to be $\omega'$, and assume $\alpha < \omega < \omega'$ for all ordinals $\alpha$. The indicator of an element $x$ will be denoted $H(x)$. An indicator is called proper if it does not contain $\omega'$. If $A$ is a submodule of $M$ and $\alpha$ is an ordinal or $\omega$, the $\alpha$th relative Ulm invariant will be denoted $f(\alpha, M, A)$.

In [5], (announced in [4]), Warfield defined a $T$-module as a module that can be defined in terms of generators and relations in such a way that the only relations are of the form $px = 0$ or $px = y$. A summand of a $T$-module is called a $T^*$-module. A module $M$ has torsion free rank one if, for any two elements $x$ and $y$ of infinite order in $M$, there are nonzero elements $r$ and $s$ in $R$ such that $rx = sy$. Warfield has shown that if $M$ is a $T$-module, it is either a torsion module, or a direct sum of modules of torsion free rank one.

A subset $X$ of a module $M$ is a decomposition basis if $[X]$ is the free module on $X$, $M/[X]$ is torsion, and $h(\sum r_ix_i) = \min\{h(r_i, x_i)\}$, for $r_i \in R$, $x_i \in X$. (Here $h(x)$ denotes the height of $x$.) It has been shown by Warfield that a $T^*$-module $M$ has a decomposition basis $X$ such that $[X]$ is nice in $M$.

In §2, we define the concept of stability, which is stronger than niceness. The invariant $S(e, M)$ defined in [2] is generalized. These concepts are used in §3 to generalize Warfield's theorem [5, Theorem 5.2].

2. Stability and invariants.

Definition. Let $M$ be a module and $A$ be a submodule. $A$ is called stable in $M$ if:

(i) $A$ is nice in $M$;

(ii) every coset $m + A$ of infinite order contains an element $x$ such that $p'x$
is proper with respect to $A$ for $i = 0, 1, 2, \ldots$. Such an element $x$ is called strongly proper with respect to $A$.

We list without proof several elementary properties.

(A) A nice submodule $A$ is stable in $M$ exactly if $\mu(M/A) = (\mu M + A)/A$
for all proper indicators $\mu$.

(B) If $x$ is strongly proper with respect to $A$, then $H(x + a) = \min\{H(x), H(a)\}$ for all $a \in A$.

(C) Direct summands are stable.

(D) Let $N_i$ be a summand of $M_i$ for $i \in I$. Then $\bigoplus_{i \in I} N_i$ is stable in $\bigoplus_{i \in I} M_i$ if and only if each $N_i$ is stable in $M_i$.

(E) Let $A$ and $B$ be submodules of $M$ with $A \subseteq B \subseteq M$. If $A$ is stable in $M$ and $B/A$ is stable in $M/A$, then $B$ is stable in $M$.

(F) Let $M$ have torsion free rank one and let $x \in M$. Then $[x]$ is stable in $M$.

(G) Let $M$ be a $T^*$-module. Then $M$ has a decomposition basis $X$ such that, for each $Y \subseteq X$, $[Y]$ is stable in $M$.

Let $M$ be a module and $\mu = \{\alpha_0, \alpha_1, \ldots\}$ be a proper indicator. The following were defined in [2]:

$$\mu M = \{m \in M : H(m) \geq \mu\};$$

$$\mu^* M = \{m \in \mu M : \text{for infinitely many } i, h(p^i m) > \alpha_i\}.$$

Let $A$ be a submodule of $M$. Then

$$M(p, A) = \{m \in M : \text{there is } k > 0 \text{ such that } p^k m \in A\}.$$

Let $\mu^*(M, A) = \mu M \cap (\mu^* M + M(p, A))$. Then $\mu M/\mu^*(M, A)$ is a vector space over $R/\mu R$ if $\mu$ does not contain $\infty$, and is a free $R$-module if $\mu$ contains $\infty$. In either case, the rank $r(\mu M/\mu^*(M, A))$ is defined. We may now define the relative $S$-invariants.

**Definition.** Let $M$ be a module, $A$ a submodule and $e$ an equivalence class of indicators. Then

$$S(e, M, A) = \sup_{\mu \in e} \{r(\mu M/\mu^*(M, A))\}.$$

These invariants are called relative $S$-invariants.

Let $\mu = \{\alpha_0, \alpha_1, \ldots\}$ be an indicator and $i \geq 0$.

Then $\mu_i$ is defined by $\mu_i = \{\alpha_i, \alpha_{i+1}, \ldots\}$.

**Lemma 2.1.** The map $\phi: \mu M/\mu^*(M, A) \to \mu_i M/\mu_i^*(M, A)$, defined by

$$\phi(x + \mu^*(M, A)) = p^i x + \mu_i^*(M, A),$$

is a monomorphism.

The proof is similar to Lemma 1 of [2].

3. A Generalization of Warfield's Theorem. The following theorem generalizes Theorem 5.2 of Warfield [5].

**Theorem 3.1.** Let $M$ and $N$ be modules, with submodules $A$ and $B$ respec-
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...tively, subject to the following conditions.

(i) $A$ is stable in $M$ and $B$ is stable in $N$.

(ii) $M/A$ and $N/B$ are $T^*$-modules.

(iii) $f(\alpha, M, A) = f(\alpha, N, B)$ for all ordinals $\alpha$.

(iv) $S(e, M, A) = S(e, N, B)$ for all equivalence classes $e$ of proper indicators.

(v) There is a height preserving isomorphism $\psi: A \to B$.

Then $\psi$ can be extended to an isomorphism $\phi: M \to N$.

**Proof.** By (G) of the previous section and (ii), $M/A$ has a decomposition basis $X$ such that $X'$ is stable in $M/A$ for every $X' \subseteq X$. Likewise $N/B$ has a decomposition basis $Y$ with the analogous property. We may assume, by the proof of Lemma 5.1 of [5], that $X$ has the following property: Whenever $f(\alpha, M, A)$ is infinite, then $f(\alpha, M, A \oplus [X])$. Likewise, we may assume that whenever $f(\alpha, N, B)$ is infinite, then $f(\alpha, N, B \oplus [Y])$. It is also permissible to assume that there is a bijection $\beta: X \to Y$ such that $H(x) = H(\beta x)$ for all $x \in X$.

We interrupt the proof in order to show that a "one step extension" is possible.

**Lemma 3.2.** Let the situation of Theorem 3.1 be given and let $x \in X$. Then $\psi$ can be extended to a height preserving isomorphism $\zeta: A \oplus [x] \to B \oplus [\beta x]$ such that:

(i) $A \oplus [x]$ is stable in $M$ and $B \oplus [\beta x]$ is stable in $N$.

(ii) $M/(A \oplus [x])$ and $N/(B \oplus [\beta x])$ are $T^*$-modules.

(iii) $f(\alpha, M, A \oplus [x]) = f(\alpha, N, B \oplus [\beta x])$ for all $\alpha$.

(iv) $S(e, M, A \oplus [x]) = S(e, M, B \oplus [\beta x])$ for all $e$.

**Proof.** Since $[A, x]/A$ is free, $A \oplus [x]$ is really a direct sum, so the map $\zeta$ can be defined by $\zeta(a + rx) = \varphi a + r\beta x$. $\zeta$ is trivially height preserving. Property (i) follows from (E), while (ii) is a known property of $T^*$-modules. For (iii), we use the following well-known formula.

$$f(\alpha, M, A \oplus [x]) = f(\alpha, M, A) - 1$$

if $f(\alpha, M, A)$ is finite, and there is $r$ such that $h(rx) = \alpha$, but $h(prx) > \alpha + 1$.

$$f(\alpha, M, A \oplus [x]) = f(\alpha, M, A)$$

otherwise.

A standard argument shows the following. If $e$ is a class of indicators, then

$$\{ x \in X : H(x) \in e \} = S(e, M, A).$$

With this observation, (iv) is immediate.

**Conclusion of proof of Theorem 3.1.** Let $\theta: A \oplus [X] \to B \oplus [Y]$ be the unique isomorphism which extends $\psi$ and $\beta$. $\theta$ is a height preserving isomorphism, $A \oplus [X]$ is nice in $M$ and $B \oplus [Y]$ is nice in $[Y]$. $M/(A \oplus [X])$ and $N/(B \oplus [Y])$ are torsion $T^*$-modules. Because of the choice of $X$ and $Y$ and Lemma 3.2, the relative Ulm invariants are equal:

$$f(\alpha, M, A \oplus [X]) = f(\alpha, N, B \oplus [Y]).$$
By the analogue to Ulm's theorem for totally projective groups, (see Walker [3, Theorem 2.8]), $\theta$ can be extended to an isomorphism $\phi: M \to N$.

REFERENCES

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