CONDITIONS FOR COMMUTATIVE SEMIGROUPS TO HAVE NONTRIVIAL HOMOMORPHISMS INTO NONNEGATIVE (POSITIVE) REALS

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Abstract. Let $S$ be a commutative cancellative semigroup of finite rank (the free rank of the quotient group of $S$ is finite). We give a necessary and sufficient condition for $S$ to have a nontrivial homomorphism into the additive semigroup of nonnegative (positive) reals. We also give a counter-example in the case of infinite rank.

1. Introduction and statement of Theorem. Throughout this note "semigroup" means a commutative cancellative semigroup and its operation is written additively. $R_0$ (resp. $R_+$) denotes the additive semigroup of nonnegative (resp. positive) reals. It is a fundamental fact that any $N$-semigroup is homomorphic into $R_+$ [4], [5]. What is a necessary and sufficient condition under which a semigroup has a nontrivial homomorphism into $R_0$ (resp. $R_+$)? This problem is treated in [3] in some special cases. In this note we solve the problem under a certain finiteness condition.

Let $S$ be a semigroup and let $\hat{S}$ be its quotient group. The rank of $S$ means the free rank of $\hat{S}$ and is denoted by rank $S$. $\mathbb{Z}_+$ denotes the set of all positive integers. The theorem we are going to prove is

Theorem. Let $S$ be a commutative cancellative semigroup of finite rank. Then:

(1) $\text{Hom}(S, R_0) \neq 0$ if and only if $S$ is not a group.
(2) $\text{Hom}(S, R_+) \neq \emptyset$ if and only if $S$ satisfies the condition

(*) for any $a, b \in S$ there exists $n \in \mathbb{Z}_+$ such that $lna \nmid lb$ for all $l \in \mathbb{Z}_+$.

In the last section we give an example; a semigroup $S$ of infinite rank satisfying condition (*) but $\text{Hom}(S, R_0) = 0$ (a fortiori $\text{Hom}(S, R_+) = \emptyset$).

2. Reduction. Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Q}$ (resp. $\mathbb{R}$) be the rational (resp. real) number field. Assume rank $S = r < \infty$. Let $\hat{S}$ be the quotient group of $S$ and let $\iota_2: S \to \hat{S}$ be the injection. By the isomorphism $\hat{S} \otimes_{\mathbb{Z}} Q \simeq Q^r$, we identify $\hat{S} \otimes_{\mathbb{Z}} Q$ and $Q^r$. Let $\iota_2: \hat{S} \to \hat{S} \otimes_{\mathbb{Z}} Q$ be the
canonical mapping. Let \( i = i_1 \circ i_2 \) and \( C_S = \{ a \in \mathbb{Q}' \mid na \in \iota(S) \text{ for some } n \in \mathbb{Z}_+ \} \).

**Lemma 1.** (1) \( \text{Hom}(S, \mathbb{R}_0) \cong \text{Hom}(C_S, \mathbb{R}_0) \).
(2) \( \text{Hom}(S, \mathbb{R}_+) \cong \text{Hom}(C_S, \mathbb{R}_+) \).

**Proof.** \( f \in \text{Hom}(S, \mathbb{R}_0) \) is extended uniquely to \( \tilde{f} \in \text{Hom}(\tilde{S}, \mathbb{R}) \) in the natural way. \( \tilde{f} \) is extended uniquely to \( \hat{f} \in \text{Hom}(\tilde{S} \otimes \mathbb{Z} \mathbb{Q}, \mathbb{R}) \) by \( \hat{f}(s \otimes q) = \tilde{f}(s)q \). It is easy to see \( \hat{f} \geq 0 \) on \( C_S \), and, moreover, \( \hat{f} > 0 \) on \( C_S \) if \( f > 0 \). This proves the lemma.

**Lemma 2.** \( S \) is a group if and only if \( C_S \) is a group.

**Proof.** If \( S \) is a group, \( \iota(S) \) is a group. It is easy to prove that \( C_S \) is a group. Conversely, assume \( C_S \) is a group. Then for any \( s \in S \) there exists \( x \in C_S \) such that \( s \otimes 1 + x = 0 \). By the definition of \( C_S \), \( nx = s' \otimes 1 \in \iota(S) \) for some \( n \in \mathbb{Z}_+ \) and some \( s' \in S \). Hence \( (ns + s') \otimes 1 = 0 \). Therefore there exists \( m \in \mathbb{Z}_+ \) such that \( m(ns + s') = 0 \). This shows \( (mn - 1)s + ms' \) is the inverse element of \( s \). Thus every element of \( S \) is invertible, and then \( S \) is a group.

The proof of the following lemma is routine and we omit it.

**Lemma 3.** \( S \) satisfies condition \( (*) \) in §1 if and only if \( C_S \) satisfies the condition

\[
(*)
\bigcap_{n=1}^{\infty} (nx + C_S) = \emptyset \quad \text{for all } x \in C_S.
\]

By the definition of \( C_S \), \( C_S \) is a subsemigroup of \( \mathbb{Q}' \) satisfying that \( nx \in C_S \) for some \( n \in \mathbb{Z}_+ \) implies \( x \in C_S \). That is, \( C_S \) is a convex cone in \( \mathbb{Q}' \).

3. **Convex cones in \( \mathbb{Q}' \) and \( \mathbb{R}' \).** Let \( S \) be a convex cone in \( \mathbb{Q}' \). \( \mathbb{Q}' \) is considered as a subset of \( \mathbb{R}' \) naturally. Let \( \tilde{S} \) be the closure of \( S \) in \( \mathbb{R}' \) (we always consider the usual topology of \( \mathbb{R}' \)). Then \( \tilde{S} \) is a closed convex cone in \( \mathbb{R}' \).

**Lemma 4.** \( \text{Hom}(S, \mathbb{R}_0) \cong \text{Hom}(\tilde{S}, \mathbb{R}_0) \).

**Proof.** \( f \in \text{Hom}(S, \mathbb{R}_0) \) can be extended to \( \tilde{f} \in \text{Hom}(\tilde{S}, \mathbb{R}) \), and \( \tilde{f} \) can be extended to \( F \in \text{Hom}(\tilde{Q}', \mathbb{R}) \) since \( \mathbb{R} \) is a divisible abelian group (see [2] for example). From the isomorphism \( \text{Hom}(\tilde{Q}', \mathbb{R}) \cong \mathbb{R}' \), it follows that \( F \) (consequently \( f \)) is continuous. Therefore \( f \) is extended uniquely to \( \tilde{f} \in \text{Hom}(\tilde{S}, \mathbb{R}_0) \).

**Lemma 5.** Let \( A \) be a convex set of \( \mathbb{Q}' \) and let \( A' \) be the convex hull of \( A \) in \( \mathbb{R}' \). Then \( A' \cap \mathbb{Q}' = A \).

**Lemma 6.** \( S \) is a group if and only if \( \tilde{S} \) is a group.

**Proof.** It is easy to see that \( \tilde{S} \) is a group if \( S \) is a group. Conversely, assume that \( \tilde{S} \) is a group. Since \( \tilde{S} \) is an \( \mathbb{R} \)-vector subspace of \( \mathbb{R}' \), we may assume...
\( \tilde{S} = \mathbb{R}' \). Since \( S \) is dense in \( \mathbb{R}' \), it follows that \( S' = \mathbb{R}' \). By Lemma 5, \( S = S' \cap Q' = Q' \), thus \( S \) is a group.

\( I(\tilde{S}) \) denotes the subgroup of \( \tilde{S} \) of all invertible elements of \( \tilde{S} \). Obviously \( I(\tilde{S}) \) is an \( \mathbb{R} \)-vector subspace of \( \mathbb{R}' \).

**Lemma 7.** \( I(\tilde{S}) \cap S = \{ a \in S | \cap_{n=1}^{\infty} (na + S) \neq \emptyset \} \).

**Proof.** Assume that \( b \in \cap_{n=1}^{\infty} (na + S) \neq \emptyset \). Then \( b - na \in S \) for all \( n \in \mathbb{Z}_+ \). Let \( c_n = b/n - a \). Then \( c_n \in S \). Since \( c_n \to -a \) as \( n \to \infty \), it follows that \( -a \in S \). This shows \( a \in I(\tilde{S}) \cap S \). Conversely let \( a \in I(\tilde{S}) \cap S \). Let \( L \) be the hyperplane through \( -a \) perpendicular to the line through \( a \) and the origin. \( L \cap S \) is convex and dense in \( L \cap \tilde{S} \). By Lemma 8, which we state below, there exists \( b \in L \cap S \) such that \( (b - na)/(n + 1) \in L \cap S \) for all \( n \in \mathbb{Z}_+ \). Hence \( b - na \in S \) for all \( n \in \mathbb{Z}_+ \), that is, \( b \in \cap_{n=1}^{\infty} (na + S) \neq \emptyset \). This completes the proof.

**Lemma 8.** Let \( A \) be a convex set of \( Q' \) and let \( \overline{A} \) be the closure of \( A \) in \( \mathbb{R}' \). Let \( a \in \overline{A} \cap Q' \). Then there exists \( b \in A \) such that every rational point of the open segment joining \( a \) and \( b \) is contained in \( A \).

**Proof.** Let \( A' \) be the convex hull of \( A \) in \( \mathbb{R}' \). If \( A' \) has an inner point, we can choose an element \( b \) of \( A \) which is an inner point of \( A' \) since \( A \) is dense in \( A' \). Then it is known that the open segment joining \( a \) and \( b \) is contained in \( A' \) (see [1, Proposition 16, p. 54]). Hence every rational point of the segment is contained in \( A' \cap Q' = A \). If \( A' \) has no inner point, then \( A \) does not contain \( r + 1 \) points in general position. Therefore \( A \) is contained in a hyperplane of \( Q' \). Thus the proof is reduced to the case of dimension \( r - 1 \). Repeating this, we may only prove the lemma in the case of \( r = 1 \), but in this case the assertion of the lemma is clear. The proof is completed.

4. **Proof of Theorem.** The necessity of the conditions of the Theorem is clear. By virtue of Lemmata 1, 2, 4 and 6, in order to prove (1) of the Theorem we may show for a convex cone \( S \) in \( Q' \) that

(1') if \( \tilde{S} \) is not a group, then \( \text{Hom}(\tilde{S}, \mathbb{R}_0) \neq 0 \).

By virtue of Lemmata 1, 3 and 7, in order to prove (2) of the Theorem we may show for a convex cone \( S \) in \( Q' \) that

(2') if \( I(\tilde{S}) \cap S = \emptyset \), then \( \text{Hom}(S, \mathbb{R}_+) \neq \emptyset \).

Let \( \eta : \mathbb{R}' \to \mathbb{R}'/I(\tilde{S}) \) be the canonical surjection to the quotient space. Then \( \eta(\tilde{S}) \) is a closed convex cone in \( \mathbb{R}'/I(\tilde{S}) \) satisfying \( I(\eta(\tilde{S})) = 0 \). Then \( \eta(\tilde{S}) \) is convex strictly at the origin, hence contained strictly in a half-space of \( \mathbb{R}'/I(\tilde{S}) \) except the origin. Therefore, there exists a functional \( \varphi \) of \( \mathbb{R}'/I(\tilde{S}) \) such that \( \varphi(x) > 0 \) for all \( x \in \eta(\tilde{S}) \setminus \{0\} \). Let \( f = (\varphi \circ \eta)_{\tilde{S}} \), then \( f \in \text{Hom}(\tilde{S}, \mathbb{R}_0) \) and \( f(a) > 0 \) for all \( a \in S \setminus I(\tilde{S}) \). If \( \tilde{S} \) is not a group, then \( \tilde{S} \setminus I(\tilde{S}) \neq \emptyset \), hence \( f \neq 0 \). Therefore \( \text{Hom}(\tilde{S}, \mathbb{R}_0) \neq 0 \), this proves (1'). If \( \tilde{S} \) satisfies \( I(\tilde{S}) \cap S = \emptyset \), then \( f|_S > 0 \). Therefore \( \text{Hom}(S, \mathbb{R}_+) \neq \emptyset \); this proves (2'). The proof of the Theorem is complete.
5. **Counterexample in the case of infinite rank.** In the case of infinite rank, condition (\(*\)) in the Theorem does not imply \(\text{Hom}(S, R_q) \neq 0\) as the following example \(S\) shows. \(S\) is constructed as a subsemigroup of the free group \(\oplus_1^{\infty} \mathbb{Z}\) of countable rank. Let \(T_n\) be an \(n \times n\)-matrix:

\[
T_n = \begin{pmatrix}
    \binom{n-1}{n-1} & \binom{n-1}{n-2} & \cdots & \binom{n-1}{0} \\
    \binom{n-2}{n-2} & \binom{n-2}{n-3} & \cdots & \binom{n-2}{0} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & \cdots & \binom{0}{0}
\end{pmatrix},
\]

where \((\cdot)\) denotes the binomial coefficients.

For \(x = (x_1, x_2, \ldots, x_i, \ldots) \in \bigoplus_1^{\infty} \mathbb{Z}\) or for \(x = (x_1, x_2, \ldots, x_m) \in \bigoplus_1^{\infty} \mathbb{Z}\) with \(m > n\), we define \(\tau_n(x) = (x_1, x_2, \ldots, x_n)\) and \(\tau_n(x) = x_m\). Set \(S(n) = \{x \in \bigoplus_1^{\infty} \mathbb{Z} | T_n \tau_n(x) > 0\}\), and \(\tau_n(x) = 0\) for \(m > n\), where \(T_n \tau_n(x) > 0\) means that all components of \(T_n \tau_n(x)\) are positive. Now we define \(S = \bigcup_{m=1}^{\infty} S(n) \subset \bigoplus_1^{\infty} \mathbb{Z}\).

**\(S\) is a semigroup:** We define \(\bar{S}(n) = \{x \in \bigoplus_1^{\infty} \mathbb{Z} | T_n \tau_n(x) > 0\}\), and \(\tau_n(x) = 0\) for \(m > n\). By a simple calculation using the formula

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-2} + \cdots + \binom{r-1}{r-1},
\]

we find \(\bar{S}(n) \subset \bar{S}(m)\) if \(m > n\). Hence \(S(m) + S(n) \subset S(m) + \bar{S}(m) \subset S(m)\). This shows that \(S\) is closed with respect to addition, so \(S\) is a semigroup.

**\(S\) satisfies condition (\(*\)):** Let \(a \in S(p)\) and \(b \in S(q)\). If \(p > q\), then clearly \(l(b - a) \not\in S\) for all \(l \in \mathbb{Z}_+.\) If \(p < q\), then \(\pi_p(a) > 0\), and \(\pi_p(a) = 0\) for \(p' > p\). Hence \(\pi_p(T_q \tau_q(a)) = \pi_p(a) > 0\). If we choose \(n \in \mathbb{Z}_+\) such that \(n > \pi_p(T_q \tau_q(b))/\pi_p(a)\), then

\[
\pi_p(T_q \tau_q(l(b - na))) = l(\pi_p(T_q \tau_q(b)) - n\pi_p(a)) < 0
\]

for all \(l \in \mathbb{Z}_+.\) This implies \(l(b - na) \not\in S\) for all \(l \in \mathbb{Z}_+.\) Thus we see in either case that \(S\) satisfies condition (\(*\)).

**Hom(\(S, R_q\)) = 0:** We define \(\bar{S} = \bigcup_{m=1}^{\infty} \bar{S}(n)\). Since \(f \in \text{Hom}(S(n), R_q)\) is extended uniquely to \(\bar{f} \in \text{Hom}(\bar{S}(n), R_q)\), \(\varphi \in \text{Hom}(S, R_q)\) is extended uniquely to \(\bar{\varphi} \in \text{Hom}(\bar{S}, R_q)\). Hence it suffices to prove \(\text{Hom}(\bar{S}, R_q) = 0\). Let \(n_0 \in \mathbb{Z}_+\) and let \(x \in \bigoplus_1^{\infty} \mathbb{Z}\) satisfy that \(\pi_{n_0}(x) > 0\) and \(\pi_n(x) = 0\) for \(n > n_0\). Then we easily have \(T_m \tau_m(x) > 0\) for a sufficiently large \(m\), hence \(x \in \bar{S}(m) \subset \bar{S}\). Therefore, for any \(a \in \bar{S}\) (say \(a \in \bar{S}(p)\)) there exists \(b \in S\) (any \(b \in S(p + 1)\) is available) such that \(la|b\) in \(\bar{S}\) for all \(l \in \mathbb{Z}_+.\) From this it follows immediately that \(\text{Hom}(\bar{S}, R_q) = 0\).

**References**


