ON THE DIAMETERS OF COMPACT RIEMANN SURFACES

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ABSTRACT. We derive an inequality relating the diameter and the length of a simple closed geodesic on a compact Riemann surface.

1. Introduction. Let $S$ be a compact Riemann surface of genus $g \geq 2$. Let $G$ be the Fuchsian group representing $S$. The metric on $S = U/G$ is the Poincaré metric, which is induced from the Poincaré metric $|dz|/y$ in the upper half plane $U$. This is the only metric we use throughout this paper. Let $\alpha$ be a simple closed geodesic on $S$. If we vary the conformal structure on $S$ so that the length of $\alpha$ goes to zero, then the surface $S$ goes to the boundary of Teichmüller space. This kind of deformation was justified by Keen [2] using the existence of collars on Riemann surfaces.

In [4] Mumford proved a general compactness theorem for Fuchsian groups of the first kind under the hypotheses that all groups $G$ considered are torsion free and $U/G$ is compact. These additional conditions were removed by Bers [1]. Along the lines of Mumford's proof, he derived an inequality relating the diameter and a shortest simple closed geodesic on a compact Riemannian manifold. In Riemann surface theory it can be read as follows. Let $S$ be a compact Riemann surface of genus $g$. Let $d$ be the diameter of $S$ and let $m$ be the length of a shortest simple closed (nontrivial) geodesic on $S$; then

$$md < 2 \text{area}(S).$$

In this paper, we shall find a sharper inequality and an inequality in the reverse direction, from which we conclude that if $S$ goes to the boundary of Teichmüller space, then the diameter of $S$ will go to infinity. As a matter of fact, the inequality we found is true for any simple closed geodesic (not necessarily of shortest length). Moreover, we allow the group $G$ representing $S$ to have elliptic elements.

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2. Definitions and statements of results. Let $G$ be a finitely generated Fuchsian group of the first kind with signature $\sigma = (g, n; \nu_1, \nu_2, \ldots, \nu_n)$, where $g$ and $n$ are positive integers or zero, the $\nu_i$ are integers or the symbol...
∞, and 2 < ν_1 < ν_2 < ⋯ < ν_n < ∞. It is known that the area of \( G \) is given by

\[
A = \text{area}(G) = \int_{U/G} y^{-2} \, dx \, dy
\]

\[
= 2\pi(2g - 2 + n - \nu_1^{-1} - \cdots - \nu_n^{-1}) > 0.
\]

Note that \( U/G \) is compact if and only if \( n = 0 \) or \( \nu_n < \infty \); we call such signature of compact type. We denote by \( \text{diam}(G) \) the diameter of \( U/G \). The diameter is finite if and only if \( U/G \) is compact.

We denote by \( X(\sigma) \) the set of conjugacy classes \([G]\) of Fuchsian groups \( G \) with signature \( \sigma \). Mumford’s compactness theorem states that if \( \sigma \) is of compact type, then the subset of \( X(\sigma) \) corresponding to groups \( G \) so that all geodesics on \( U/G \) have length > \( c \) (a constant) is compact.

Let \( S = U/G \) be a compact Riemann surface. Let \( \alpha \) be a minimal geodesic of length \( a \) on \( S \). By a band \( B \) around \( \alpha \) of radius \( b \) we mean the union of all geodesics on \( S \) perpendicular to \( \alpha \), where each is of length \( b \). Since the Poincaré metric on \( U \) is given by

\[
A(z)|az| = 2(z - \bar{z})^{-1}|dz|,
\]

the ray \( \theta = \theta_0, 0 < \theta_0 < \pi/2 \), is of distance \( \log|\csc \theta_0 + \cot \theta_0| \) to the imaginary axis \( \theta = \pi/2 \). Lifting \( B \) up to the upper half plane \( U \) with \( \alpha \) lying on the imaginary axis, an easy computation shows that \( B \) is a region bounded by the curves \( \rho = 1, \rho = e^a, \theta = \csc^{-1}(\cosh(b)) \) and \( \theta = \pi - \csc^{-1}(\cosh(b)) \). One easily sees that the noneuclidean area of \( B \) is \( 2a \sinh(b) \).

Now we state the main results.

**Theorem.** Let \( G \) be a Fuchsian group so that \( U/G \) is compact with \( d = \text{diam}(G) \) and \( A = \text{area}(G) \). Then we have

(1) \[ 2r \sinh(d) > A, \]

where \( r \) is the length of a simple closed geodesic on \( S \); and

(2) \[ 2 \sinh(m/4) \, d < A, \]

where \( m \) is the length of a shortest simple closed geodesic on \( S \).

The following corollaries follow from the previous theorem together with Mumford’s compactness theorem.

**Corollary 1.** Let \( \sigma \) be the type \((g, 0)\). If \( G \) is on the boundary of \( \text{Teichmüller space} \ X(\sigma) \), then \( U/G \) is not compact.

With Bers’ generalization of Mumford’s theorem, we have

**Corollary 2 (Bers [1]).** Let \( \sigma \) be of compact type. The subset of \( X(\sigma) \) corresponding to groups \( G \) with \( \text{diam}(G) < c < \infty \) is compact.

3. **Proof of the theorem.** Let \( \beta \) be a simple closed geodesic on \( S = U/G \) with length \( r \). Let \( p \) be any point on \( \beta \) which is not a fixed point of \( G \). Form
the Dirichlet region $D$ of $G$ in $U$ with center at $p = ie^{\pi/2}$ and $\beta$ lying on the imaginary axis of $U$. Then $D$ is contained in a strip bounded by the curves $\rho = 1$ and $\rho = e^\rho$. We construct a band $B$ around $\beta$ of radius $d$. Lifting $B$ to $U$, $B$ is the region bounded by the curves $\rho = 1$, $\rho = e^\rho$, $\theta = \csc^{-1}(\cosh(d))$ and $\theta = \pi - \csc^{-1}(\cosh(d))$. Since any point $q$ on $S$ is of distance at most $d$ to the center $p = ie^{\pi/2}$, $q$ is of distance at most $d$ to $\beta$. Thus $D \subset B$ and hence,

$$A = \text{area}(G) = \text{area}(D) \leq \text{area}(B) = 2\pi \sinh(d).$$

This proves (1).

Let $\delta$ be a minimal geodesic realizing the diameter of $S$ with endpoints $x$ and $y$. Let $B'$ be the band around $\delta$ of radius $m/4$. We first prove that no two such geodesics of $B'$ meet. Suppose $\delta_1$, $\delta_2$ meet at the point $w$. Let $z_1$, $z_2$ be the points on $\delta$ from which $\delta_1$, $\delta_2$ originate, and $e$ be the distance from $z_1$ to $z_2$ along $\delta$. Then we can go from $x$ to $y$ by going from $x$ to $z_1$ on $\delta$, following $\delta_1$, then $\delta_2$, and going from $z_2$ to $y$ on $\delta$. This has length $< d - e + m/2$, and since $\delta$ is the shortest path from $x$ to $y$, $d < d - e + m/2$, i.e., $e < m/2$. But then $\delta_1$, $\delta_2$ and the part of $\delta$ between $z_1$ and $z_2$ is a closed curve $\tau$ of length at most $m$. $\tau$ is certainly not homotopic to zero, for otherwise $\tau$ would be lifted to a triangle in the upper half plane $U$ with two right interior angles. Moreover, $\tau$ has corners and so it is not a geodesic. Therefore there is a closed geodesic freely homotopic to $\tau$ of length $< m$, which is impossible.

This shows that, with the choice of the radius $m/4$, the whole band $B'$ is contained in $S$. So we have

$$2 \sinh(m/4)d = \text{area}(B') \leq \text{area}(S) = A,$$

and the theorem is proved.

REFERENCES


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