ISOMETRIES OF QUASITRIANGULAR OPERATOR ALGEBRAS

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Abstract. Let \((P_n)\) be an increasing sequence of finite rank projections on a separable Hilbert space. Assume \(P_n\) converges strongly to the identity operator \(I\). The quasi triangular operator algebra determined by \((P_n)\) is defined to be the set of all bounded linear operators \(T\) for which
\[
\lim_{n \to \infty} \| (I - P_n) TP_n \| = 0.
\]

In this note we prove that two quasitriangular algebras are unitarily equivalent if, and only if, there exists a unital linear isometry mapping one algebra onto the other.

This note adds one further piece of information about isomorphisms of quasi triangular operator algebras to the facts already gathered in [7]. If \(\mathcal{P} = \{P_n\}_{n=1}^{\infty}\) is a sequence of finite rank projections increasing to the identity and acting on a separable Hilbert space \(\mathcal{H}\), then the quasi triangular algebra \(\mathcal{T}(\mathcal{P})\) is defined to be the set of all bounded operators \(T\) in \(\mathcal{B}(\mathcal{H})\) such that \(\|P_n^{-1} TP_n\| \to 0\) as \(n \to \infty\). Two quasi triangular algebras, \(\mathcal{T}(\mathcal{P})\) and \(\mathcal{T}(\mathcal{R})\) are said to be isometric if there exists a linear isometry \(\phi\) mapping \(\mathcal{T}(\mathcal{P})\) onto \(\mathcal{T}(\mathcal{R})\) such that \(\phi(I) = I\). In [7] it is shown that the following three conditions are equivalent:

(a) \(\mathcal{T}(\mathcal{P})\) is algebraically isomorphic to \(\mathcal{T}(\mathcal{R})\);

(b) there exist positive integers \(n_0\) and \(m_0\) such that \(\dim(P_{n_0+k}) = \dim(R_{m_0+k})\) for all \(k > 0\);

(c) \(\mathcal{T}(\mathcal{P})\) is unitarily equivalent to \(\mathcal{T}(\mathcal{R})\).

Here, we add one further equivalent condition to the list:

(d) \(\mathcal{T}(\mathcal{P})\) is isometric to \(\mathcal{T}(\mathcal{R})\).

First, we prove a lemma concerning extensions of isometries which may be of independent interest. \(\mathcal{C}(\mathcal{H})\) denotes the set of compact operators.

Lemma. Let \(\psi_0 : \mathcal{C}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\) be a linear isometry with the property that, for each pair of vectors \(x, y \in \mathcal{H}\),
\[
\sup \{ |\langle \psi_0(T)x, y \rangle| : T \in \mathcal{C}(\mathcal{H}) \text{ and } \|T\| = 1 \} = \|x\| \|y\|.
\]

Then there exists a unique extension of \(\psi_0\) to a linear isometry, \(\psi\), mapping \(\mathcal{B}(\mathcal{H})\) into \(\mathcal{B}(\mathcal{H})\).
Proof. For each pair of vectors \( x, y \in \mathcal{H} \) define a linear functional \( \phi^0_{x,y} \) on \( \mathcal{C}(\mathcal{H}) \) by \( \phi^0_{x,y}(T) = \langle \phi_0(T)x, y \rangle \), for all \( T \in \mathcal{C}(\mathcal{H}) \). The hypothesis on \( \phi_0 \) asserts that \( \|\phi^0_{x,y}\| = \|x\| \|y\| \). By the Hahn-Banach theorem, there exists an extension, \( \phi_{x,y} \), of \( \phi^0_{x,y} \) to \( \mathcal{B}(\mathcal{H}) \) such that \( \|\phi_{x,y}\| = \|x\| \|y\| \). By [2, Theorem 3], \( \phi_{x,y} \) has a unique decomposition \( \phi_{x,y} = \alpha + \beta \), where \( \alpha \) is ultraweakly continuous; \( B(\mathcal{C}(\mathcal{H})) = 0 \); and \( \|\phi_{x,y}\| = \|x\| \|y\| \). Since \( \|\phi_{x,y}\| = \|\phi^0_{x,y}|C(\mathcal{H})\| = \|\alpha\| \), we obtain \( \beta = 0 \); i.e. \( \phi_{x,y} \) is ultraweakly continuous. This already proves the uniqueness of extensions, for if \( \psi_1 \) and \( \psi_2 \) are both linear isometries which extend \( \phi_0 \), then the linear functionals \( A \rightarrow \langle \psi_1(A)x, y \rangle \) and \( A \rightarrow \langle \psi_2(A)x, y \rangle \) are both ultraweakly continuous and agree on \( \mathcal{C}(\mathcal{H}) \). Since \( \mathcal{C}(\mathcal{H}) \) is ultraweakly dense in \( \mathcal{B}(\mathcal{H}) \), \( \langle \psi_1(A)x, y \rangle = \langle \psi_2(A)x, y \rangle \), for all \( x, y \in \mathcal{H} \). Hence \( \psi_1 = \psi_2 \).

The ultraweak continuity of the \( \phi_{x,y} \) implies that, for each \( A \in \mathcal{B}(\mathcal{H}) \), \( (x, y) \rightarrow \phi_{x,y}(A) \) is a bilinear form on \( \mathcal{H} \) bounded by \( \|A\| \). Consequently there exists an operator \( \psi(A) \) in \( \mathcal{B}(\mathcal{H}) \) such that \( \phi_{x,y}(A) = \langle \psi(A)x, y \rangle \), for all \( x, y \in \mathcal{H} \), and \( \|\psi(A)\| \leq \|A\| \). It is routine to check that \( \psi \) is linear and that \( \psi|C(\mathcal{H}) = \phi_0 \). To prove that \( \psi \) is an isometry, let \( E_n \) be a sequence of finite rank projections such that \( E_n \uparrow I \) in the strong topology. Let \( A \in \mathcal{B}(\mathcal{H}) \) and let \( A_n = AE_n \), for each \( n \). For each pair \( x, y \in \mathcal{H} \),

\[
\langle \psi(A)x, y \rangle = \phi_{x,y}(A) = \lim_{n} \phi_{x,y}(A_n) = \lim_{n} \langle \phi_0(A_n)x, y \rangle.
\]

Hence, given \( \epsilon > 0 \), there exist unit vectors \( x \) and \( y \) such that \( \|\phi_0(A_n)x\| - \epsilon = \|A_n\| - \epsilon \). Thus, \( \|\phi_0(A_n)\| > \|A_n\| \), for all \( n \). Since \( \|A_n\| \rightarrow \|A\| \), we have \( \|\phi_0(A)\| = \|A\| \), for all \( A \in \mathcal{B}(\mathcal{H}) \), and \( \psi \) is an isometry.

Remark. If we drop the condition \( \sup |\langle \phi_0(T)x, y \rangle| = \|x\| \|y\| \), then the uniqueness is no longer true. Indeed, let \( \text{id} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) be the identity map and \( 0 \) be the zero map. Also, let \( \phi \) be any linear map of \( \mathcal{B}(\mathcal{H}) \) into itself such that \( \|\phi\| \leq 1 \) and \( \phi|C(\mathcal{H}) = 0 \). Then after a suitable identification of \( \mathcal{H} \oplus \mathcal{H} \) with \( \mathcal{H} \), \( \text{id} \oplus 0 \) and \( \text{id} \oplus \phi \) are two distinct linear isometries of \( \mathcal{B}(\mathcal{H}) \) which agree on \( C(\mathcal{H}) \). The proof of existence also breaks down, as the ultracontinuity of the \( \phi_{x,y} \) depends on the fact that the functional achieves its norm on \( C(\mathcal{H}) \).

Let \( \mathcal{L}(\mathcal{H}) \) denote \( \mathcal{L}(\mathcal{H}) \cap \mathcal{L}(\mathcal{H})^* \) and note that \( \mathcal{L}(\mathcal{H}) = \{ T | \|P_n T - TP_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \} \) [6, Proposition 16].

Theorem. The quasitriangular algebras \( \mathcal{L}(\mathcal{H}) \) and \( \mathcal{L}(\mathcal{R}) \) are unitarily equivalent if and only if they are isometric.

Proof. It is clear that if \( \mathcal{L}(\mathcal{H}) \) is unitarily equivalent to \( \mathcal{L}(\mathcal{R}) \) then they are isometric. For the converse, suppose that \( \phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{R}) \) is a linear isometry of \( \mathcal{L}(\mathcal{H}) \) onto \( \mathcal{L}(\mathcal{R}) \) such that \( \phi(I) = I \). By [4, Lemma 8], \( \phi \) preserves adjoints on \( \mathcal{L}(\mathcal{H}) \) and hence maps \( \mathcal{L}(\mathcal{H}) \) onto \( \mathcal{L}(\mathcal{R}) \). By Theorem 8 in [4] the restriction \( \phi_0 \) of \( \phi \) to \( \mathcal{L}(\mathcal{H}) \) is a \( C^* \)-isomorphism (Jordan isomorphism). Because \( \mathcal{L}(\mathcal{H}) \) and \( \mathcal{L}(\mathcal{R}) \) are irreducible \( C^* \)-algebras, we may conclude that \( \phi_0 \) is either a \( \ast \)-isomorphism or a \( \ast \)-anti-isomorphism (see [8, Theorem 6.4] or [5, Theorem 2.6]).
Actually, $\phi_0$ must be a $*$-isomorphism; the assumption that $\phi_0$ is a $*$-anti-isomorphism leads to a contradiction, as follows. Let $\mathcal{S} = \{e_1, e_2, \ldots\}$ be an orthonormal basis of $\mathcal{H}$ such that each $R_n$ is the projection on the span of $\{e_1, \ldots, e_k\}$, where $k = \dim R_n$. Let $t: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be the transpose with respect to $\mathcal{S}$: $t$ is defined by the condition $\langle t(A) e_i, e_j \rangle = \langle A e_i, e_j \rangle$, for all $A \in \mathcal{B}(\mathcal{H})$. Note that $t$ is an anti-isomorphism and a linear isometry of $\mathcal{B}(\mathcal{H})$ onto itself. Observe, further, that $t(2 \mathcal{D}(\mathcal{H})) = 2 \mathcal{D}(\mathcal{H})$. Indeed, by the choice of $\mathcal{S}$, $t(R_n) = R_n$, for all $n$. Then

$$\|AR_n - R_n A\| = \|t(AR_n - R_n A)\| = \|t(R_n) t(A) - t(A) t(R_n)\| = \|R_n t(A) - t(A) R_n\|,$$

for all $n$. Thus, $A \in 2 \mathcal{D}(\mathcal{H})$ if, and only if, $t(A) \in 2 \mathcal{D}(\mathcal{H})$. Now let $\psi = t \circ \phi$. So, $\psi$ is a linear isometry of $2 \mathcal{T}(\mathcal{F})$ into $\mathcal{B}(\mathcal{H})$ and $\psi_0 = t \circ \phi_0$ is a $*$-isomorphism of $2 \mathcal{D}(\mathcal{F})$ onto $2 \mathcal{D}(\mathcal{H})$. Since $2 \mathcal{D}(\mathcal{F})$ and $2 \mathcal{D}(\mathcal{H})$ both contain $\mathcal{C}(\mathcal{K})$, $\psi_0$ is implemented by a unitary operator $V$. ([1, Corollary 3, p. 20, Theorem 1.3.4].) By the lemma above, $\psi$ is also implemented by $V$; hence the image of $2 \mathcal{T}(\mathcal{F})$ under $\psi$ is a quasitriangular algebra $2 \mathcal{T}(S)$, where $S = (S_n) = (VP_n V^{-1})$. Therefore, $t$ maps $2 \mathcal{T}(\mathcal{H})$ onto $2 \mathcal{T}(S)$. Let $U$ be the unilateral shift with respect to $\mathcal{S}$, viz. $U e_n = e_{n+1}$, for all $n$. Then the backward shift, $U^*$, is easily seen to be an element of $2 \mathcal{T}(S)$; hence $U = t(U^*) \in 2 \mathcal{T}(S)$. But this says that $U$ is a quasitriangular operator and Halmos has proven the contrary in [3].

Therefore, we know that $\phi_0$ is a $*$-isomorphism of $2 \mathcal{D}(\mathcal{F})$ onto $2 \mathcal{D}(\mathcal{H})$. But then, as above, it follows that $\phi_0$ is implemented by a unitary operator $V$, and, from the lemma, that $\phi$ is implemented by $V$. Thus $2 \mathcal{T}(\mathcal{F})$ is unitarily equivalent to $2 \mathcal{T}(\mathcal{H})$.

REFERENCES

5. __________, Transformation of states in operator theory and dynamics, Topology 3 (1965), 177-198.
7. __________, Quasitriangular operator algebras, Pacific J. Math. 64 (1976), 543-549.

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