PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 65, Number 2, August 1977

ON AUTOMORPHISMS OF L.C. GROUPS

JUSTIN PETERS

ABSTRACT. The left regular representation λ of a locally compact group G generates a W^*-algebra $\mathfrak{R}(\lambda)$, and each topological automorphism $\tilde{\alpha}$ of G has a natural extension to an automorphism $\tilde{\alpha}$ of $\mathfrak{R}(\lambda)$. It is proved that an automorphism β of $\mathfrak{R}(\lambda)$ is of the form $\beta = \tilde{\alpha}$ for $\alpha \in \operatorname{Aut}(G)$ iff β leaves a certain cone in $\mathfrak{R}(\lambda)$ invariant.

Recently a number of important results concerning automorphisms of W^*-algebras have been obtained (see e.g. [2], [4]), and in some cases these results have analogues in automorphisms of locally compact groups, or can be applied directly to yield facts about group automorphisms [6]. Let λ denote the left regular representation of a locally compact group G ($\lambda(x)$, $x \in G$, operates on an $L^2(G)$-function by left translation by x^{-1}) and $\mathfrak{R}(G)$ be the double commutant of $\lambda(G) = \{\lambda(x): x \in G\}$, or the W^*-algebra generated by $\lambda(G)$. There is a natural imbedding of $\operatorname{Aut}(G) \hookrightarrow \operatorname{Aut} \mathfrak{R}(G)$: to $\alpha \in \operatorname{Aut}(G)$ there corresponds a unique $\tilde{\alpha} \in \operatorname{Aut} \mathfrak{R}(G)$ satisfying $\tilde{\alpha}\lambda(x) = \lambda(\alpha(x))$. In this note we characterize those automorphisms of $\mathfrak{R}(G)$ which come from automorphisms of G (via the imbedding) as the set of automorphisms of $\mathfrak{R}(G)$ which leave a certain cone fixed. Also, a connection between automorphisms of $\mathfrak{R}(G)$ and the measure algebra $\mathfrak{M}(G)$ is mentioned.

If A is a Banach *-algebra, we denote by A'^+ the positive cone in the dual A' given by $\{f \in A': f(a^*a) \geqslant 0, a \in A\}$. There is also a cone $A^+ = \{a \in A: f(a) \geqslant 0, f \in A'^+\}$ in A. A ray in A'^+ is a set of the form R^+f, $f \in A'^+$ ($R^+ = \{r \in R: r \geqslant 0\}$). A linear functional $f \in A'^+$ is said to lie on an extreme ray if $f = g + h$, $g, h \in A'^+$, implies $g, h \in R^+f$. The following proposition is an extension of a well-known theorem of Kelley and Vaught [5].

PROPOSITION. *Let A be a commutative Banach *-algebra with continuous involution, and $B \subset A$ a dense subalgebra. Suppose $\{e_i\} \subset B \cap A^+$ is a (not necessarily bounded) approximate identity for B. Let $f \in A'^+$; then f lies on an extreme ray if and only if θf is multiplicative for some θ, $0 \neq \theta \in R^+$.*

PROOF. Let $f \in A'^+$ lie on an extreme ray and choose $b \in B$ such that

Received by the editors January 26, 1977.
AMS (MOS) subject classifications (1970). Primary 22D45.
Key words and phrases. Automorphism, W^*-algebra, Fourier algebra, measure algebra, group character.

$\|b^*b\| < 1$ and $f(b^*b) > 0$. Define $g \in A'^+$ by $g(y) = f(yb^*b)$. Clearly g is positive since $g(y^*y) = f((yb)^*(yb)) \geqslant 0$. Likewise $(f - g) \in A'^+$, for $(f - g)(y^*y) = f(z^*z)$, where $z = \sum_{n=0}^{\infty}\binom{1/2}{n}y(-b^*b)^n$. (Observe that if we adjoined an identity to A in the canonical way we could write

$$z = y\sqrt{1 - b^*b} = y\sum_{n=0}^{\infty}\binom{1/2}{n}(-b^*b)^n.\Big)$$

Since we can write $f = g + (f - g)$ with g, $(f - g) \in A'^+$, it follows from the assumption on f that $g = \mu f$, for some μ, $0 \leqslant \mu < \infty$. But $g(e_i) = f(e_i b^*b) \neq 0$ for sufficiently large i, so $\mu > 0$. This implies $\lim f(e_i)$ exists, and, in fact,

$$\lim f(e_i) = \mu^{-1}\lim g(e_i) = \mu^{-1}\lim f(e_i b^*b) = \mu^{-1}f(b^*b) < \infty.$$

Let

$$\theta = \mu/f(b^*b) = 1/\lim f(e_i).$$

Note that θ does not depend on b. Then $\theta g = \mu\theta f$ and $\mu = \theta f(b^*b)$. Thus

$$\theta g(y) = \theta f(b^*b)\theta f(y), \qquad y \in A,$$

or

$$\theta f(b^*by) = \theta f(b^*b)\theta f(y), \qquad y \in A.$$

Observe now that this last equation is valid if in place of b^*b we have a^*a, $a \in A$, where $f(a^*a) = 0$. For in that case $|f(a^*ay)|^2 \leqslant f(a^*a)f((ay)^*(ay)) = 0$. Since any $x \in B^2$ can be written as a linear combination of elements of the form b^*b, $\|b^*b\| < 1$, we have $\theta f(xy) = \theta f(x)\theta f(y)$, $x \in B^2$, $y \in A$. Finally, using that $B^2 \subset A$ is dense along with the continuity of f, we have that $\theta f(xy) = \theta f(x)\theta f(y)$ holds for all $x, y \in A$.

The converse is easy. For let $f \in A'^+$ be multiplicative and suppose $f = g + h$, $g, h \in A'^+$. Let N_f (resp., N_g) be the null space of f (resp., g). If $x \in N_f$, then $f(x^*x) = f(x^*)f(x) = 0$, hence $g(x^*x) = 0$. But then $|g(x)|^2 \leqslant g(x^*x) = 0$, so $N_g \supset N_f$. This means $g = \mu f$ for some μ, $0 \leqslant \mu < \infty$, and, consequently, f lies on an extreme ray. □

We apply the foregoing proposition by taking A to be $A(G)$, the Fourier algebra of a locally compact group G. Recall that $A(G)$ is a Banach*- algebra of continuous functions on G vanishing at infinity under pointwise multiplication with complex conjugation as involution. If $C_{00}(G)$ denotes the continuous functions with compact support on G, then $B = C_{00}(G) \cap A(G)$ is dense in $A(G)$. Given a compact set $k \subset G$ there is a function $a_k \in B$, $0 \leqslant a_k \leqslant 1$, $a_k|k = 1$ [3, 3.2]. If for each compact set $k \subset G$ we choose such an a_k, order the k's by inclusion and set $e_k = a_k^2 = a_k^* a_k$, then $\{e_k\} \subset B \cap A^+$ constitutes an (unbounded) approximate identity for B, so the hypotheses of the proposition are satisfied.

Now $A(G)' = \mathfrak{R}(G)$, and $\Delta(A(G))$, the spectrum of $A(G)$, is identified with G, or, more properly, $\lambda(G)$ (acting by pointwise evaluation) [3, 3.34]. Let $P = A(G)'^+$. The cone P is not to be confused with the cone of positive operators in $\mathfrak{R}(G)$: indeed, $\lambda(x)$ ($x \in G$) is clearly in P, but $\lambda(x)$ is not even hermitian if $x \neq x^{-1}$. Applying the above proposition, we find that the set of extreme rays of P is precisely $\{R^+\lambda(x): x \in G\}$.

By an automorphism of a W^* algebra we mean, of course, a *-automorphism, and this will automatically be norm-preserving.

COROLLARY. *An automorphism β of $\mathfrak{R}(G)$ comes from an automorphism of G if and only if $\beta P = P$.*

PROOF. P has a metrizable topology and so is well capped [1, 30.19]; hence by the Choquet theory P is the closed convex hull of its extreme rays. If $\beta = \tilde{\alpha}$, $\alpha \in \mathrm{Aut}(G)$, β maps the set of extreme rays of P onto itself, thus $\beta P = P$. Suppose, conversely, that $\beta P = P$. If $\beta\lambda(x) = S + T$, $S, T \in P$, then $\lambda(x) = \beta^{-1}S + \beta^{-1}T$. Thus $\beta^{-1}S = \mu\lambda(x)$, some $0 \leqslant \mu < \infty$, or $S = \mu\beta\lambda(x)$, which means $\beta\lambda(x)$ lies on an extreme ray. Since $\|\beta\lambda(x)\| = 1$, we must have $\beta\lambda(x) = \lambda(y)$, for some $y \in G$. Setting $y = \alpha(x)$ we obtain a map $\alpha: G \to G$. An easy argument now shows $\alpha \in \mathrm{Aut}(G)$. □

Instead of looking at the extreme rays of a certain cone in $\mathfrak{R}(G)$ we could, of course, consider the extreme points of the unit ball in $\mathfrak{R}(G)$, $\mathrm{Ext}(\mathfrak{R}(G)_1)$. But any $\beta \in \mathrm{Aut}\,\mathfrak{R}(G)$ must map $\mathrm{Ext}(\mathfrak{R}(G)_1)$ onto itself, so in place of $\mathfrak{R}(G)_1$ we might take $\mathfrak{M}(G)_1$, where $\mathfrak{M}(G)$ is the measure algebra. Now $A(G) \subset C_0(G)$, the continuous functions on G vanishing at infinity, so we can view $\mathfrak{M}(G) = C_0(G)'$ as a subalgebra of $\mathfrak{R}(G)$. If $\mathfrak{M}(G)$ is given its own norm (and not the norm it inherits from $\mathfrak{R}(G)$), then $\mathrm{Ext}(\mathfrak{M}(G)_1) = \{e^{i\theta}\delta_x: \theta \in R, x \in G\}$, δ_x being the point mass at x. Indeed, if $\mu \in \mathfrak{M}(G)_1$, $\|\mu\| = 1$, is nonatomic, then there are nonzero $\mu_1, \mu_2 \in \mathfrak{M}(G)$, support$(\mu_i)$ $\subset$ support(μ), $i = 1, 2$, satisfying support$(\mu_1) \cap$ support(μ_2) is a $|\mu|$-null set, and $\mu = \mu_1 + \mu_2$ with $\|\mu_1\| + \|\mu_2\| = \|\mu\| = 1$. Setting $V_i = \|\mu_i\|^{-1}\mu_i$, $i = 1, 2$, we have $\mu = \|\mu_1\|V_1 + \|\mu_2\|V_2$, so μ is not extreme. On the other hand, $e^{i\theta}\delta_x \in \mathrm{Ext}(\mathfrak{M}(G)_1)_1$. In fact, viewing $\mathfrak{M}(G) \subset \mathfrak{R}(G)$, $e^{i\theta}\delta_x$ belongs to the larger unit ball $\mathfrak{R}(G)_1$, and $e^{i\theta}\delta_x \in \mathrm{Ext}(\mathfrak{R}(G)_1)$ by [7, 1.6.4].

PROPOSITION. *$\beta \in \mathrm{Aut}\,\mathfrak{R}(G)$ restricts to an isometric automorphism of $\mathfrak{M}(G)$ if and only if $\beta = \tilde{\alpha} \circ \tilde{\gamma}$, where $\beta \in \mathrm{Aut}(G)$ and γ is a group character.*

PROOF. Consider first that a group character acts as a *-automorphism of the group algebra $L^1(G)$: defining $(\gamma f)(x) = \gamma(x)f(x)$, $f \in L^1(G)$, we have that $(\gamma f) * (\gamma g) = \gamma(f * g)$, $f, g \in L^1(G)$, and $(\gamma f)^* = \gamma f^*$, where $f^*(x) = \Delta(x)^{-1}f(x^{-1})^-$ is the involution in $L^1(G)$. It is clear that γ extends to an isometric *-automorphism of the measure algebra; for $\mu \in \mathfrak{M}(G)$, we have $d(\gamma\mu)(x) = \gamma(x)\,d\mu(x)$. Let γ act on $\mathfrak{R}(G)$ by defining $(\tilde{\gamma}T)g = \gamma(T(\bar{\gamma}g))$, $T \in \mathfrak{R}(G)$, $g \in L^2(G)$, where γ acts by pointwise multiplication on $L^2(G)$; i.e., $\tilde{\gamma}T = \gamma T\bar{\gamma}$. To see this define a *-automorphism of $\mathfrak{R}(G)$, first note for $T \in \mathfrak{R}(G)$, $g, h \in L^2(G)$ that

$$
\begin{aligned}
((\tilde{\gamma}T)^*g, h) &= (g, \tilde{\gamma}Th) = (g, \gamma T\bar{\gamma}h) = (\bar{\gamma}g, T\bar{\gamma}h) \\
&= (T^*\bar{\gamma}g, \bar{\gamma}h) = (\gamma T^*\bar{\gamma}g, h) = (\tilde{\gamma}T^*g, h).
\end{aligned}
$$

To show $\tilde{\gamma}$ is multiplicative, observe, if $f \in L^1(G)$, $\tilde{\gamma}\lambda(f) = \lambda(\gamma f)$. (We use the same notation for the left regular representation of $L^1(G)$, which acts by left convolution on $L^2(G)$, as we do for the left regular representation of G, since the former is just the Bochner-integrated form of the latter.) So for $f, g \in L^1(G)$,

$$\begin{aligned} \tilde{\gamma}(\lambda(f)\lambda(g)) &= \tilde{\gamma}\lambda(f * g) = \lambda(\gamma(f * g)) \\ &= \lambda((\gamma f) * (\gamma g)) = \lambda(\gamma f)\lambda(\gamma g) = \tilde{\gamma}\lambda(f)\tilde{\gamma}\lambda(g). \end{aligned}$$

Then use that $\{\lambda(f): f \in L^1(G)\} \subset \mathfrak{R}(G)$ is strongly dense and the fact that multiplication in $\mathfrak{R}(G)$ is jointly strongly continuous on bounded subsets to see $\tilde{\gamma}$ is multiplicative on $\mathfrak{R}(G)$.

Suppose now $\beta \in \operatorname{Aut} \mathfrak{R}(G)$ restricts to an isometric automorphism of $\mathfrak{M}(G)$. Then β maps the unit ball of $\mathfrak{M}(G)$ onto itself, hence $\operatorname{Ext}(\mathfrak{M}(G)_1)$ is mapped onto itself. Thus $\beta\delta_x = e^{i\theta}\delta_{\alpha(x)}$, for some $\theta \in R$, and $\alpha: G \to G$. If we set $|\beta|\delta_x = \delta_{\alpha(x)}$, it is clear that $|\beta|$ is multiplicative on $\operatorname{Ext}(\mathfrak{M}(G)_1)$, hence $\alpha \in \operatorname{Aut}(G)$. Set $\beta\delta_x = \gamma(x)\delta_{\alpha(x)}$. A simple argument shows γ is multiplicative, hence is a group character. Thus β agrees with $\tilde{\alpha} \circ \tilde{\gamma}$ on $\lambda(G)$, and since $\lambda(G)$ generates $\mathfrak{R}(G)$, we must have $\beta = \tilde{\alpha} \circ \tilde{\gamma}$.

References

1. G. Choquet, *Lectures on analysis*, Vol. II, Benjamin, New York, 1969.
2. A. Connes, *Almost periodic states and factors of type* III_1, J. Functional Analysis **16** (1974), 415–445.
3. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.
4. U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37** (1975), 271–283.
5. J. Kelley and R. Vaught, *The positive cone in Banach algebras*, Trans. Amer. Math. Soc. **74** (1953), 44–55.
6. J. Peters and T. Sund, *Automorphisms for locally compact groups*, Univ. Oslo Inst. Math. Preprint Series, no. 14, 1976.
7. S. Sakai, *C*-algebras and W*-algebras*, Springer-Verlag, Berlin and New York, 1971.
8. M. Takesaki and N. Tatsuuma, *Duality and subgroups*, Ann. of Math. (2) **93** (1971), 344–364.

Department of Mathematics, Iowa State University, Ames, Iowa 50011