AN INDIVIDUAL ERGODIC THEOREM

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Abstract. An individual ergodic theorem is proved for a linear operator $T$ on $L_1$ of a finite measure space which satisfies certain norm conditions.

Derriennic and Lin [5] showed by an example that given an $\varepsilon > 0$ there exists a positive linear operator $T$ on $L_1$ of a finite measure space, with $T1 = 1$ and $\|T^n\|_1 = 1 + \varepsilon$ for all $n \geq 1$, and a function $f$ in $L_1$ such that the individual ergodic limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

does not exist almost everywhere on a certain measurable subset of positive measure.

In this paper, however, we shall prove the following individual ergodic theorem.

Theorem. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $L_p = L_p(X, \mathcal{F}, \mu)$, $1 < p < \infty$, the usual Banach spaces. Let $T$ be a bounded linear operator on $L_1$ and $\tau$ its linear modulus in the sense of Chacon and Krengel [4]. Assume the conditions:

1. $\sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \right\|_1 < \infty$,
2. $\sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \right\|_{\infty} < \infty$.

Then, for any $f \in L_{\infty}$, the ergodic limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

exists for almost all $x \in X$.

Proof. Let $T^*$ and $\tau^*$ denote the adjoint operators of $T$ and $\tau$, respectively. Since $|T^* f| \leq \tau^* |f|$ (cf. [1]) and $\int \tau^* |f| \, d\mu = \int (\tau 1) |f| \, d\mu \leq \|	au\|_\infty \|f\|_1$ for all
$f \in L_\infty$, $T^*$ and $\tau^*$ can be extended to bounded linear operators $S$ and $\sigma$ on $L_1$, respectively. It is easily seen that

$$S^* = T^* \quad (on \ L_\infty) \quad and \quad \sigma^* = \tau^* \quad (on \ L_\infty).$$

Hence it follows that the linear modulus of $S$ is $\sigma$ and that

$$\sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \sigma^i \right\|_1 = \sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \right\|_\infty < \infty.$$

We now define two functions $u$ and $v$ in $L_\infty$ by the relations:

$$u(x) = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \chi(x) \quad (x \in X)$$
and

$$v(x) = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \chi(x) \quad (x \in X).$$

It may be readily seen that $\tau^* u \geq u$ and $\tau v \geq v$. Thus if we set

$$s(x) = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^* u(x) \quad (x \in X)$$
and

$$t(x) = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau v(x) \quad (x \in X),$$

then

(3) \quad $\tau^* s = s$ and $\tau t = t$.

Let $Y = \{x : s(x) > 0\}$ and $Z = X - Y$. Then, by [5] and [7], we have:

(i) if $f \in L_1(Z)$ implies $\tau f \in L_1(Z)$,
(ii) if $f \in L_1(Z)$ then $\lim_n \|/(1/n) \sum_{i=0}^{n-1} \tau f\|_1 = 0$,
(iii) $\lim_n \|\tau f\|_1/n = 0$ for every $f \in L_1$.

We shall now divide the proof of the theorem into several steps, since it is rather long.

Step I. If $f \in L_\infty(Z)$ then $\lim_n (1/n) \sum_{i=0}^{n-1} T^f(x) = 0$ for almost all $x \in X$.

To prove this, we may and do assume without loss of generality that $0 < f < 1$ on $Z$, and it is enough to show that the function $h$ defined by

$$h(x) = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau f(x) \quad (x \in X)$$

satisfies $h(x) = 0$ almost everywhere on $X$. In fact, we notice that $0 < h \in L_\infty(Z)$ (c $L_1(Z)$), by (2) and (i), and that $\tau h \geq h$. Hence (ii) implies that

$$\|h\|_1 < \lim_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau h \right\|_1 = 0,$$

and therefore $h(x) = 0$ almost everywhere on $X$.

Step II. For any $f \in L_\infty$ the limit $\lim_n (1/n) \sum_{i=0}^{n-1} T^f(x)$ exists for almost all $x \in Y$. 

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To see this, we first notice that

$$\int |Tf| s \, d\mu \leq \int (|f| \tau s) \, d\mu = \int |f| \tau^* s \, d\mu = \int |f| s \, d\mu$$

for all $f \in L_1$. Thus $T$ may be regarded as a linear contraction operator on $L_1(Y, s \, d\mu)$, since $L_1 = L_1(X, \mathcal{F}, \mu)$ is a dense subspace of $L_1(Y, s \, d\mu)$. Therefore we can apply Chacon's general ratio ergodic theorem [2], [3] to $T$ to infer that, for any $f \in L_\infty = L_\infty(X, \mathcal{F}, \mu) \subset L_1(Y, s \, d\mu)$, the limit

$$\lim_{n} \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) = \lim_{n} \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) / \sum_{i=0}^{n-1} \tau^i(x)$$

exists almost everywhere on $Y \cap \{x: \tau(x) > 0\}$. On the other hand it is immediate from the definition of $\tau$ that, for any $f \in L_\infty$,

$$\lim_{n} \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) = 0$$

almost everywhere on $\{x: \tau(x) = 0\}$. Thus Step II is established.

**Step III.** If $f \in L_\infty$ and $\lim_{n} \| (1/n) \sum_{i=0}^{n-1} T^i f \|_1 = 0$, then $\lim_{n} \| (1/n) \sum_{i=0}^{n-1} T^i f(x) = 0$ for almost all $x \in X$.

To prove this, define

$$f_0(x) = \lim_{n} \sup \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) \right| \quad (x \in X).$$

Clearly, $0 < f_0 \in L_\infty$, and it follows from Step II that $f_0(x) = 0$ for almost all $x \in Y$. For each $k \geq 1$, put

$$f_k = \frac{1}{k} \sum_{i=0}^{k-1} T^i f.$$

By an easy computation we then have

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f - \frac{1}{n} \sum_{i=0}^{n-1} T^i f_k \right| = \left| \frac{1}{n} \left( T^n - I \right) \sum_{i=1}^{k-1} \left( 1 - \frac{i}{k} \right) T^{i-1} f \right|.$$

To see that

$$\lim_{n} \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) - \frac{1}{n} \sum_{i=0}^{n-1} T^i f_k(x) \right| = 0$$

for almost all $x \in X$, define

$$h(x) = \sum_{i=1}^{k-1} \left( 1 - \frac{i}{k} \right) T^{i-1} f(x) \quad (x \in X)$$

and

$$\tilde{h}(x) = \lim_{n} \sup \left| \frac{1}{n} \tau^n h(x) \right| \quad (x \in X).$$

Then we have
for almost all $x \in X$, and furthermore

$$
\tau \tilde{h}(x) = \tau \left( \limsup_n \frac{1}{n} \tau^n h(x) \right) \geq \limsup_n \frac{1}{n} \tau^{n+1} h(x) = \tilde{h}(x)
$$

for almost all $x \in X$. Here we can apply Step II to $\tau$ instead of $T$ and obtain that $\tilde{h}(x) = 0$ for almost all $x \in Y$. Therefore $\tilde{h} \in L_\infty(Z) \subset L_1(Z)$, and so

$$
\|\tilde{h}\|_1 < \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \|\tilde{h}\| = 0,
$$

by (ii). Consequently, $\tilde{h}(x) = 0$ for almost all $x \in X$.

It then follows that, for almost all $x \in X$,

$$
\limsup_n \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) - \frac{1}{n} \sum_{i=0}^{n-1} T^i f_k(x) \right| = 0 \quad (k > 1).
$$

Thus, for almost all $x \in X$,

$$
f_0(x) = \limsup_n \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f_k(x) \right| \leq \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i |f_k|(x)
$$

$$
< \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i f_k^1(x) + \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i f_k^2(x),
$$

where $f_k^1 = |f_k|_Z$ and $f_k^2 = |f_k|_Y$. By this and the argument in Step I, we obtain that

$$
f_0(x) \leq \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i f_k^2(x)
$$

for almost all $x \in X$, because $f_k^1 \in L_\infty(Z)$.

Let us put, for each $k > 1$,

$$
\tilde{f}_k(x) = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i f_k^2(x) \quad (x \in X).
$$

It follows that $0 < \tilde{f}_k \in L_\infty$ and that $\tau^i \tilde{f}_k > \tilde{f}_k$. Hence, writing $g_k^1 = \tilde{f}_k 1_Z$ and $g_k^2 = \tilde{f}_k 1_Y$, and applying (ii) to $g_k^1 \in L_\infty(Z) \subset L_1(Z)$, we get

$$
\|\tilde{f}_k\|_1 \leq \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i g_k^1
$$

$$
< \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i g_k^2 + \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i g_k^3,
$$

$$
= \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i g_k^2 < \sup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \|g_k^2\|_1.
$$
On the other hand, applying Step II to \( \tau \) instead of \( T \), we observe that

\[
g_k^2(x) = \lim_n \left( \frac{1}{n} \sum_{i=0}^{n-1} \tau_j f_k^2(x) \right) 1_\gamma(x)
\]

for almost all \( x \in X \). Hence, by Fatou’s lemma,

\[
\| g_k^2 \|_1 \leq \liminf_n \left\| \left( \frac{1}{n} \sum_{i=0}^{n-1} \tau_j f_k^2 \right) 1_\gamma \right\|_1 \\
\leq \left\{ \sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau_j \right\|_1 \right\} \| f_k^2 \|_1.
\]

By this and the fact that

\[
\| g_k \|, = \lim_k \| f_k \|, = 0,
\]

we obtain that \( \lim_k \| g_k^2 \|_1 = 0 \), and hence that \( \lim_k \| f_k \|_1 = 0 \). Since \( 0 < f_0(x) < f_k(x) \) for almost all \( x \in X \), it must follow that \( \| f_0 \|_1 = 0 \). So \( f_0(x) = 0 \) almost everywhere on \( X \), and this establishes Step III.

**Step IV.** For any \( f \in L_\infty \) there exists a function \( g \in L_\infty \) such that \( Tg = g \) and

\[
\lim_n \|(1/n)\Sigma_{i=0}^{n-1} T^i(f - g)\|_1 = 0.
\]

To prove this, let \( f \in L_\infty \) be given. Then, by (2) and Theorem IV.8.9 in [6], the set \( \{(1/n)\Sigma_{i=0}^{n-1} T^i f: n > 1\} \) is weakly sequentially compact in \( L_1 \), and by (iii), \( \lim_n \| T^nf \|_1/n < \lim_n \| n^{-1} T^n f \|_1/n = 0 \). Hence, a well-known mean ergodic theorem (cf. Theorem VIII.5.1 in [6]) shows that there exists a function \( g \in L_1 \) such that \( Tg = g \) and

\[
\lim_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i f - g \right\|_1 = 0.
\]

Clearly \( g \in L_\infty \), by condition (2), and hence Step IV is established.

**Step V.** We shall now conclude the proof of the theorem as follows.

Let \( f \in L_\infty \) be given, and using Step IV, write \( f = g + h \) where \( Tg = g \in L_\infty \) and \( h \in L_\infty \) satisfies \( \lim_n \|(1/n)\Sigma_{i=0}^{n-1} T^i h\|_1 = 0 \). Then, by Step III, we observe that

\[
\lim_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) - g(x) \right\|_1 = 0
\]

for almost all \( x \in X \). Hence the proof is completed.

**Remark.** In [8], an analogous result is proved for a strongly continuous one-parameter semigroup \( \{T_t\}_{0 < t < \infty} \) of positive linear operators on \( L_1 \) of a finite measure space.

**Example.** We shall construct a positive linear operator \( T \) on \( L_1 \) of a finite measure space which satisfies the following norm conditions:
(7) \[ \sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i \right\|_1 < \infty \quad \text{and} \quad \sup_n \| T^n \|_1 = \infty, \]

(8) \[ \sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i \right\|_\infty < \infty \quad \text{and} \quad \sup_n \| T^n \|_\infty = \infty. \]

Put \( a_0 = 0, a_1 = 1, a_2 = 2, a_n = 4a_{n-1} \) \((n \geq 3)\), and \( b_n = \sum_{i=0}^{n} a_i \) \((n \geq 0)\).

Write \( X = \{(n, i) : n \geq 1 \text{ and } 1 < i < b_n\} \), and let \( \mathcal{F} \) be all possible subsets of \( X \) and \( \lambda \) the measure on \((X, \mathcal{F})\) defined by \( \lambda(\{(1, 1)\}) = 1 \) and, for \( n \geq 2, \)

\[
\lambda(\{(n, i)\}) = \begin{cases} 
\frac{1}{2^n-1}, & \text{if } 1 < i < a_n, \\
\lambda(\{(n-1, i-a_n)\}), & \text{if } a_n < i < b_n.
\end{cases}
\]

Put \( \mu(\{(n, i)\}) = \left(\frac{1}{4^n}\right) \lambda(\{(n, i)\}) \). Then it follows from an easy computation that \((X, \mathcal{F}, \mu)\) is a probability measure space. Define a point mapping \( \varphi \) from \( X \) to \( X \) by the relation:

\[
\varphi((n, i)) = \begin{cases} 
(n, i + 1), & \text{if } 1 < i < b_n, \\
(n + 1, 1), & \text{if } i = b_n.
\end{cases}
\]

Then \( X = \{\varphi^n((1, 1)) : n \geq 1\} \), and thus if we set, for convenience's sake, \( \varphi^n((1, 1)) = n, \) then \( X = \{n : n \geq 0\} \). Define a positive linear operator \( S \) on \( L_1(X, \mathcal{F}, \mu) \) by the relation:

\[
Sf(n) = f(n + 1) \quad (f \in L_1(X, \mathcal{F}, \mu), n \in X).
\]

Then we have \( \sup_n \| (1/n) \sum_{i=0}^{n-1} S^i \|_1 < 4, \) since

\[
\frac{\mu(\{(n, i) : 1 < i < b_n\})}{b_n \mu(\{(n, 1)\})} < 2 \quad (n \geq 1).
\]

It is clear that \( \sup_n \| S^n \|_1 = \infty. \)

Next, let \( h \) be the function in \( L_\infty(X, \mathcal{F}, \mu) \) defined by \( h(n) = 2^{-i+1}, \) where \( b_{i-1} < n + 1 < b_i. \) Since

\[
\left( \sum_{i=0}^{b_n} h(i) \right)/(b_n + 1)h(b_n) < 4 \quad (n \geq 0),
\]

it follows that

\[
\sup_n \left\| \left( \frac{1}{n} \sum_{i=0}^{n-1} S^i h \right) / h \right\|_\infty < 4,
\]

and also that \( \sup_n \| (S^n h) / h \|_\infty = \infty. \) Hence, if we define a positive linear operator \( T \) on \( L_1(X, \mathcal{F}, h \, d\mu) \) by the relation:

\[
Tf = S(f h) / h \quad (f \in L_1(X, \mathcal{F}, h \, d\mu)),
\]

then it is easily seen that \( T \) satisfies norm conditions (7) and (8).

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Bibliography


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