AN EXAMPLE OF A SPACE WHICH IS COUNTABLY COMPACT WHOSE SQUARE IS COUNTABLY PARACOMPACT BUT NOT COUNTABLY COMPACT

LEE PARSONS

Abstract. A subspace \( P \) of \( \beta N - N \) is obtained whose square is disjoint from the graph, \( G \), of a pre-selected homeomorphism \( f: \beta N \rightarrow \beta N \) that has no fixed points. The construction is performed in such a way that, for \( X = P \cup N \), all countable subsets of \( X^2 - G \) will have a limit point in \( X^2 \).

We use the following lemma: If \( K \subset (\beta N)^2 - G \) is countably infinite, then \( |\text{cl}_{(\beta N)^2} K - G| = 2^\omega \).

We construct the space \( X \) using the technique of J. Novák [N]. In the reference cited, Novák constructs a countably compact space whose square is not countably compact. Several versions have appeared in the literature. H. Terasaka's example [T] is presented in Gillman and Jerison, Rings of continuous functions [GJ] and in Steen and Seebach, Counterexamples in topology [SS]. Novák's example was modified by Frolik in [F]. The latter version is presented by Engelking in [E], Outline of general topology.

A subspace \( P \) of \( \beta N - N \) will be obtained whose square is disjoint from the graph, \( G \), of a preselected homeomorphism \( f_\beta N \rightarrow \beta N \) that has no fixed point, but has the property that \( f^2 = f \). The notation of [GJ] is used, primarily. The construction will be performed in such a way that all countable subsets of \( X^2 - G \) will have a limit point in \( X^2 \), where \( X = P \cup N \).

Then \( X \) will be countably compact since it is homeomorphic to a closed subset of \( X^2 - G \). Moreover, \( G \cap X^2 = \{(n, f(n)): n \in N\} \) is closed in \( X^2 \) and is an infinite discrete set, so \( X^2 \) is not countably compact. But \( X^2 = (X^2 \cap G) \cup (X^2 - G) \) is the disjoint union of a countably compact subspace and a countable, clopen discrete subspace and hence is countably paracompact.

The burden of proof is borne mostly by the following

**Lemma.** If \( K \subset (\beta N)^2 - G \) is countably infinite then \( |\text{cl}_{(\beta N)^2} K - G| = 2^\omega \).

**Proof.** Suppose that \( K \subset (\beta N)^2 - G \) is countably infinite. We let \( \pi_1 \) and \( \pi_2 \) denote the projections onto the first and second factors of subsets of \( (\beta N)^2 \). If there is a point \( p \in N \) such that \( H = K \cap (\{p\} \times N) \) is infinite then \( |\text{cl} H| > |\text{cl}(\pi_2 H)| = 2^\omega \), noting that \( \text{cl} \pi_2 H = \pi_2 \text{cl} H \).

Presented to the Society, January 23, 1976; received by the editors November 7, 1975.


Key words and phrases. Countably compact, countably paracompact, extremely disconnected, \( M \)-space, pseudocompact.
\( \beta N \) is 1 so that the lemma holds in this and the analogous case, \( H(\beta N \times \{p\}) \cap K \).

We may now assume throughout (by choosing an appropriate infinite subset of \( K \)) that

\[
|K \cap (\{p\} \times \beta N)| \vee |K \cap (\beta N \times \{p\})| \leq 1
\]

\((*)\)

for all \( p \in \beta N \), and \( \pi_i K \) is infinite, \( i = 1, 2 \).

If \( K' \subset K \) is countably infinite and has the property that \( \text{cl}(f[\pi_1[K']]) \cap \text{cl}(\pi_2[K']) = \emptyset \), then it is easy to establish that \( |\text{cl}(\beta N \pi K')| = 2^{2^\omega} \) and that \( \text{cl}(\beta N \pi K' \cap G = \emptyset \) from which the lemma follows. We now devote our attention to producing such a subset of \( K \).

Every countably infinite subset of \( \beta N \) has a countably infinite subset whose topology inherited from \( \beta N \) is discrete. Now, using this fact, choose an infinite subset \( K^* \subset K \) such that \( \pi_1[K^*] \) is discrete. Then \( f[\pi_1[K^*]] \) is discrete, since \( f \) is a homeomorphism. Apply this same technique again to obtain \( K^{**} \subset K^* \), countably infinite, such that \( \pi_2[K^{**}] \) is discrete. By assumption \((*)\), \( K^{**} \) has the property that \( f[\pi_1[K^{**}]] \) and \( \pi_2[K^{**}] \) are infinite, discrete topological subspaces of \( \beta N \). Since it is a bit tedious to carry the **'s about, let us assume without loss of generality that \( K \) has the latter property to begin with.

Now cull \( K \) again. Let \( K \) be enumerated as \( \{(p_1, q_1), (p_2, q_2), \ldots \} \). Let \( i_1 = 1 \). Let \( U_1 \) be a neighborhood of \( q_1 \) which misses \( f(p_1) \) and infinitely many points of \( f[\pi_1[K]] \) and whose intersection with \( \pi_2[K] \) is \( \{q_1\} \). Now suppose \( i_1, \ldots, i_n \) are selected in such a way that \( f[\pi_1[K]] - \bigcup_{i=1}^{n} U_i \) is infinite and \( f(p_j) \notin \bigcup_{i=1}^{n} U_i \) for \( j = 1, \ldots, n \), and \( \bigcup_{i=1}^{n} U_i \cap \pi_2[K] = \{q_1, \ldots, q_n\} \). Now choose \( i_{n+1} \) so that \( f(p_{i_{n+1}}) \in f[\pi_1[K]] - \bigcup_{i=1}^{n} U_i \). Then choose \( U_{n+1} \) so that, one, it does not contain \( f(p_{i_{n+1}}) \), \( j = 1, \ldots, n+1 \); two, its intersection with \( \pi_2[K] \) is \( \{q_{n+1}\} \); and three, it misses infinitely many members of \( f[\pi_1[K]] - \bigcup_{i=1}^{n} U_i \). The inductive selection of the sequence \( \langle i_1, i_2, \ldots \rangle \) is complete. Denote by \( K' \) the subset \( \{(p_1, q_1), (p_2, q_2), \ldots \} \) of \( K \) Then \( \bigcup_{i=1}^{n} U_i \) is a neighborhood of \( \pi_2[K'] \) and \( f[\pi_1[K']] \cap \bigcup_{i=1}^{n} U_i = \emptyset \). Thus \( \text{cl}(f[\pi_1[K']]) \cap \pi_2[K'] = \emptyset \). In an exactly analogous manner, we pick an infinite subset \( K' \subset K' \) having the property that \( f[\pi_1[K']] \cap \text{cl}(\pi_2[K']) = \emptyset \). Then it follows that \( \text{cl}(f[\pi_1[K']]) \cap \pi_2[K'] = \emptyset \).

Note that \( \text{cl} \pi_1 K' = \pi_1 \text{cl} K', i = 1, 2 \), so that we actually proved:

If \( K \subset (\beta N)^2 \) is countably infinite and if \( \{p \in \beta N: (\beta N \times \{p\}) \cap K \neq \emptyset \} \) and \( \{p \in \beta N: ((p) \times \beta N) \cap K \neq \emptyset \} \) are infinite, then

\[
|\{r \in \beta N: \exists s \in \beta N, (r, s) \in \text{cl} K - G \}|
\]

\[= |\{s \in \beta N: \exists r \in \beta N, (r, s) \in \text{cl} K - G \}| = 2^\omega.
\]

Now, beginning the construction of \( X \), we index the countable subsets of \( (\beta N)^2 - G \) in type \( 2^\omega \), \( \{K_\beta\}_{\beta < 2^\omega} \). By the lemma, \( K_0 \) has a limit point which is not in \( G \cup N^2 \), \( (r_0, s_0) \). Let \( K_0 = (r_0, s_0) - N \). Inductively, suppose \( P_a, \)
α < β, are selected so that $P_α \subset P_β$ for α < γ < β and \( f[P_α] \cap P_α = \emptyset \) and $|P_α| = |α|$ if α > ω and $|P_α| < ω$ if α < ω. \( \bigcup_{α < β} P_α = \bigcup_{α < β} |α| = |β| < 2^ω \) if α > ω and is less than ω if α < ω. Consider $K_β$. Several cases arise:

(i) $\exists r_β$ such that $K_β \cap \{(r_β) \times βN\}$ is infinite.

(a) $r_β \in f(\bigcup_{α < β} P_α) \subset βN - N$. Let $P_β = \bigcup_{α < β} P_α$. In this case, P will be defined so that $K_β \subset P$ hence $K_β$ need not have a limit point in $X^2$.

(b) $r_β \notin f(\bigcup_{α < β} P_α)$. Choose $s_β \in βN - (f(\bigcup_{α < β} P_α) \cup N)$ so that $(r_β, s_β) \in cl K_β - (G \cup K_β)$. Let $P_β = \bigcup_{α < β} P_α \cup \{(r_β, s_β)\}$. [Note: the set $\bigcup_{α < β} P_α$ will be defined so that $K_β \subset P_β$ and hence $P_β$ need not have a limit point in $X^2$.]

(ii) We have an analogous case if $\exists s_β$ such that $K_β \cap (βN \times \{s_β\})$ is infinite.

(iii) If no such points exist, apply the lemma, using a simple cardinality argument, to obtain a point $(r_β, s_β)$ so that $r_β, s_β \notin f(\bigcup_{α < β} P_α) \cup N$ and $(r_β, s_β) \in cl K_β - (G \cup K_β)$. Let $P_β = \bigcup_{α < β} P_α \cup \{(r_β, s_β)\}$.

So clearly, $|P_β| = |\bigcup_{α < β} P_α| = |β|$ if α > ω and is finite otherwise. Equally clear is that $P_β \supset P_α$ for α < β.

Claim. $f[P_β] \cap P_β = \emptyset$. Let $p \in P_β$ and suppose $\exists q \in P_β$ such that $f(q) = p$. Note the following:

(i) Obviously, the inductive hypothesis guarantees that not both $p, q \in \bigcup_{α \in I} P_α$.

(ii) If $p \in \bigcup_{α < β} P_α$ and $q = r_β$, we have $f(r_β) = p$. So $f(p) = r_β$. But $r_β$ was chosen so that $r_β \notin f(\bigcup_{α < β} P_α)$.

(iii) If $p \in \bigcup_{α < β} P_α$ and $q = s_β$, $f(s_β) = p$, so $f(p) = s_β$ and we have a contradiction as above.

(iv) If $p = r_β$ and $q = s_β$, we have $f(s_β) = r_β$ so that $f(r_β) = s_β$. But this gives $(r_β, s_β) \in G$, a contradiction.

(v) If $p = s_β$ and $q = r_β$, $f(r_β) = s_β$, again a contradiction.

The claim now follows.

The inductive construction of the example is now complete. Note that $P^2$ is countably compact.

Remarks. (1) The example presented here is a partial negative answer to a question of J. Keesling, whose interest in the problem stems from research announced in [K] concerning hyperspaces. The question, to which I do not know the answer, is: If $X$ is normal and countably compact and $X^2$ is countably paracompact, is $X^2$ countably compact? R. G. Woods [Wo] has shown that $\text{CH}$ implies that if $X$ is normal, countably compact, extremally disconnected and $|C^ω(X)| = 2^ω$, then $X$ is compact. Thus the present example is not normal assuming $\text{CH}$.

(2) The example presented here also answers in the negative the following question of Morita [M]: If $X$ and $Y$ are countably compact and $X \times Y$ is an $M$-space, is $X \times Y$ countably compact? The question had been answered in the negative by Steiner [S], assuming the continuum hypothesis. An $M$-space is the quasi-perfect preimage of a metric space. Note that $X^2$ is an $M$-space: It is the free union of a countably compact space and a countably infinite discrete space. See also [Wa, pp. 188–190].
(3) An example, due to Frolik, of countably compact spaces $X$ and $Y$ whose product is pseudocompact but not countably compact is presented by Ginsburg and Saks in [GS]. Only slight modification is needed to yield a countably compact space whose square is pseudocompact but not countably compact. Similar results can be obtained from the example given by Comfort in [C].

REFERENCES


