AN EXAMPLE OF A SPACE WHICH IS COUNTABLY
COMPACT WHOSE SQUARE IS COUNTABLY
PARACOMPACT BUT NOT COUNTABLY COMPACT

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Abstract. A subspace $P$ of $\beta N - N$ is obtained whose square is disjoint from the graph, $G$, of a pre-selected homeomorphism $f: \beta N \rightarrow \beta N$ that has no fixed points. The construction is performed in such a way that, for $X = P \cup N$, all countable subsets of $X^2 - G$ will have a limit point in $X^2$. We use the following lemma: If $K \subset (\beta N)^2 - G$ is countably infinite, then $|\overline{K}(\beta N)^2| = 2^\omega$.

We construct the space $X$ using the technique of J. Novák [N]. In the reference cited, Novák constructs a countably compact space whose square is not countably compact. Several versions have appeared in the literature. H. Terasaka's example [T] is presented in Gillman and Jerison, Rings of continuous functions [GJ] and in Steen and Seebach, Counterexamples in topology [SS]. Novák's example was modified by Frolik in [F]. The latter version is presented by Engelking in [E], Outline of general topology.

A subspace $P$ of $\beta N - N$ will be obtained whose square is disjoint from the graph, $G$, of a preselected homeomorphism $f: \beta N \rightarrow \beta N$ that has no fixed point, but has the property that $f^2 = f$. The notation of [GJ] is used, primarily. The construction will be performed in such a way that all countable subsets of $X^2 - G$ will have a limit point in $X^2$, where $X = P \cup N$. Then $X$ will be countably compact since it is homeomorphic to a closed subset of $X^2 - G$. Moreover, $G \cap X^2 = \{(n, f(n)): n \in N\}$ is closed in $X^2$ and is an infinite discrete set, so $X^2$ is not countably compact. But $X^2 = (X^2 \cap G) \cup (X^2 - G)$ is the disjoint union of a countably compact subspace and a countable, clopen discrete subspace and hence is countably paracompact.

The burden of proof is borne mostly by the following

Lemma. If $K \subset (\beta N)^2 - G$ is countably infinite then $|\overline{K}(\beta N)^2| = 2^\omega$.

Proof. Suppose that $K \subset (\beta N)^2 - G$ is countably infinite. We let $\pi_1$ and $\pi_2$ denote the projections onto the first and second factors of subsets of $(\beta N)^2$. If there is a point $p \in N$ such that $H = K \cap (\{p\} \times N)$ is infinite then $|\overline{H}| \geq |\overline{\pi_2H}| = 2^\omega$, noting that $\overline{\pi_2H} = \pi_2 \overline{H}$. But $|\overline{G \cap (\{p\} \times N)}|$...
\[ \beta N \cap \{ p \times \beta N \} \cup \{ p \times \beta N \} \cap K \cap \{ p \times \beta N \} \leq 1 \]

for all \( p \in \beta N \), and \( \pi_i K \) is infinite, \( i = 1, 2 \).

If \( K' \subset K \) is countably infinite and has the property that \( \text{cl}(f[\pi_1[K^*]]) \cap \text{cl}(\pi_2[K']) = \emptyset \), then it is easy to establish that \( \text{cl}(\pi_1[K^*]) = 2^\omega \) and that \( \text{cl}(\beta N)^K \cap G = \emptyset \), from which the lemma follows. We now devote our attention to producing such a subset of \( K \).

Every countably infinite subset of \( \beta N \) has a countably infinite subset whose topology inherited from \( \beta N \) is discrete. Now, using this fact, choose an infinite subset \( K^* \subset K \) such that \( \pi_i[K^*] \) is discrete. Then \( f[\pi_1[K^*]] \) is discrete, since \( f \) is a homeomorphism. Apply this same technique again to obtain \( K^{**} \subset K^* \), countably infinite, such that \( \pi_2[K^{**}] \) is discrete. By assumption \((*)\), \( K^{**} \) has the property that \( f[\pi_1[K^{**}]] \) and \( \pi_2[K^{**}] \) are infinite, discrete topological subspaces of \( \beta N \). Since it is a bit tedious to carry the **'s about, let us assume without loss of generality that \( K \) has the latter property to begin with.

Now cull \( K \) again. Let \( K \) be enumerated as \( \{ (p_1, q_1), (p_2, q_2), \ldots \} \). Let \( i_1 = 1 \). Let \( U_1 \) be a neighborhood of \( q_1 \) which misses \( f(p_i) \) and infinitely many points of \( f[\pi_1[K]] \) and whose intersection with \( \pi_2[K] \) is \( \{ q_i \} \). Now suppose \( i_1, \ldots, i_n \) are selected in such a way that \( f[\pi_1[K]] \cap \bigcup_{i=1}^n U_i \) is infinite and \( f(p_i) \notin \bigcup_{i=1}^n U_i \) for \( j = 1, \ldots, n \), and \( \bigcup_{i=1}^n U_i \cap \pi_2[K] = \{ q_1, \ldots, q_n \} \). Now choose \( i_{n+1} \) so that \( f(p_{i_{n+1}}) \notin f[\pi_1[K]] \cap \bigcup_{i=1}^n U_i \). Then choose \( U_{n+1} \) so that, one, it does not contain \( f(p_j) \), \( j = 1, \ldots, n + 1 \); two, its intersection with \( \pi_2[K] \) is \( \{ q_{i_{n+1}} \} \); and three, it misses infinitely many members of \( f[\pi_1[K]] \cap \bigcup_{i=1}^n U_i \). The inductive selection of the sequence \( \langle i_1, i_2, \ldots \rangle \) is complete. Denote by \( K'' \) the subset \( \{ (p_i, q_i), (p_j, q_j), \ldots \} \) of \( K \). Then \( \bigcup_{i=1}^\infty U_i \) is a neighborhood of \( \pi_2[K'' \cap \bigcup_{i=1}^\infty U_i \cap f[\pi_1[K'\cap \bigcup_{i=1}^\infty U_i] = \emptyset \). Thus \( \text{cl}(\beta N)^f[\pi_1[K'\cap \bigcup_{i=1}^\infty U_i] \cap \pi_2[K'\cap \bigcup_{i=1}^\infty U_i = \emptyset \). In an exactly analogous manner, we pick an infinite subset \( K' \subset K'' \) having the property that \( f[\pi_1[K']] \cap \text{cl}(\pi_2[K']) = \emptyset \). Then it follows that \( \text{cl}(f[\pi_1[K']]) \cap \text{cl}(\pi_2[K']) = \emptyset \).

Note that \( \text{cl}(\pi_i K') = \pi_i \text{cl} K', i = 1, 2 \), so that we actually proved:

If \( K \subset (\beta N)^2 \) is countably infinite and if \( \{ p \in \beta N: (\beta N \times \{ p \}) \cap K \neq \emptyset \} \) and \( \{ p \in \beta N: \{ (p \times \beta N) \cap K \neq \emptyset \} \) are infinite, then

\[ \left| \{ r \in \beta N: \exists s \in \beta N, (r, s) \in \text{cl} K - G \} \right| = \left| \{ s \in \beta N: \exists r \in \beta N, (r, s) \in \text{cl} K - G \} \right| = 2^\omega. \]

Now, beginning the construction of \( X \), we index the countable subsets of \((\beta N)^2 - G\) in type \( 2^\omega \). (\( K_B, B < 2^\omega \). By the lemma, \( K_0 \) has a limit point which is not in \( G \cup N^2 \). Let \( P_0 = \{ r_0, s_0 \} - N \). Inductively, suppose \( P_0 \),
\( \alpha < \beta \), are selected so that \( P_{\alpha} \subset P_{\gamma} \) for \( \alpha < \gamma < \beta \) and \( f[P_{\alpha}] \cap P_{\alpha} = \emptyset \) and \( |P_{\alpha}| = |\alpha| \) if \( \alpha > \omega \) and \( |P_{\alpha}| < \omega \) if \( \alpha < \omega \). \( \cup_{\alpha < \beta} P_{\alpha} = \cup_{\alpha < \beta} |\alpha| = |\beta| < 2^\omega \) if \( \alpha > \omega \) and is less than \( \omega \) if \( \alpha < \omega \). Consider \( K_{\beta} \). Several cases arise:

(i) \( \exists \alpha \) such that \( K_{\beta} \cap (\{\alpha\} \times \beta N) \) is infinite.

(a) \( r_{\beta} \in f(\cup_{\alpha < \beta} P_{\alpha}) \subset \beta N - N \). Let \( P_{\beta} = \cup_{\alpha < \beta} P_{\alpha} \). In this case, \( P \) will be defined so that \( K_{\beta} \subseteq P \) hence \( K_{\beta} \) need not have a limit point in \( X^2 \).

(b) \( \alpha \in \cup_{\alpha < \beta} P_{\alpha} \). Choose \( s_{\beta} \in \beta N - (f[\cup_{\alpha < \beta} P_{\alpha}] \cup N) \) so that \( (r_{\beta}, s_{\beta}) \in K_{\beta} - (G \cup K_{\beta}) \). Let \( P_{\beta} = \cup_{\alpha < \beta} P_{\alpha} \cup \{(r_{\beta}, s_{\beta})\} - N \).

(ii) We have an analogous case if \( \exists \) \( \alpha \) such that \( K_{\beta} \cap (\beta N \times \{s_{\beta}\}) \) is infinite.

(iii) If no such points exist, apply the lemma, using a simple cardinality argument, to obtain a point \( (r_{\beta}, s_{\beta}) \) so that \( r_{\beta}, s_{\beta} \notin f[\cup_{\alpha < \beta} P_{\alpha}] \cup N \) and \( (r_{\beta}, s_{\beta}) \in cl K_{\beta} - (G \cup K_{\beta}) \). Let \( P_{\beta} = \cup_{\alpha < \beta} P_{\alpha} \cup \{(r_{\beta}, s_{\beta})\} \).

So clearly, \( |P_{\beta}| = |\cup_{\alpha < \beta} P_{\alpha}| = |\beta| \) if \( \alpha > \omega \) and is finite otherwise. Equally clear is that \( P_{\beta} \supset P_{\alpha} \) for \( \alpha < \beta \).

CLAIM. \( f[P_{\beta}] \cap P_{\beta} = \emptyset \). Let \( p \in P_{\beta} \) and suppose \( \exists q \in P_{\beta} \) such that \( f(q) = p \). Note the following:

(i) Obviously, the inductive hypothesis guarantees that not both \( p, q \in \cup_{\alpha \in P_{\beta}} P_{\alpha} \).

(ii) If \( p \in \cup_{\alpha < \beta} P_{\alpha} \) and \( q = r_{\beta} \), we have \( f(r_{\beta}) = p \). So \( f(p) = r_{\beta} \). But \( r_{\beta} \) was chosen so that \( r_{\beta} \notin f[\cup_{\alpha < \beta} P_{\alpha}] \).

(iii) If \( p \in \cup_{\alpha < \beta} P_{\alpha} \) and \( q = s_{\beta} \), \( f(s_{\beta}) = p \), so \( f(p) = s_{\beta} \) and we have a contradiction as above.

(iv) If \( p = r_{\beta} \) and \( q = s_{\beta} \), we have \( f(s_{\beta}) = r_{\beta} \) so that \( f(r_{\beta}) = s_{\beta} \). But this gives \( (r_{\beta}, s_{\beta}) \in G \), a contradiction.

(v) If \( p = s_{\beta} \) and \( q = r_{\beta} \), \( f(r_{\beta}) = s_{\beta} \), again a contradiction. The claim now follows.

The inductive construction of the example is now complete. Note that \( P^2 \) is countably compact.

REMARKS. (1) The example presented here is a partial negative answer to a question of J. Keesling, whose interest in the problem stems from research announced in [K] concerning hyperspaces. The question, to which I do not know the answer, is: If \( X \) is normal and countably compact and \( X^2 \) is countably paracompact, is \( X^2 \) countably compact? R. G. Woods [Wo] has shown that CH implies that if \( X \) is normal, countably compact, extremally disconnected and \( |C^*(X)| = 2^\omega \), then \( X \) is compact. Thus the present example is not normal assuming CH.

(2) The example presented here also answers in the negative the following question of Morita [M]: If \( X \) and \( Y \) are countably compact and \( X \times Y \) is an \( M \)-space, is \( X \times Y \) countably compact? The question had been answered in the negative by Steiner [S], assuming the continuum hypothesis. An \( M \)-space is the quasi-perfect preimage of a metric space. Note that \( X^2 \) is an \( M \)-space: It is the free union of a countably compact space and a countably infinite discrete space. See also [Wa, pp. 188–190].
(3) An example, due to Frolik, of countably compact spaces $X$ and $Y$ whose product is pseudocompact but not countably compact is presented by Ginsburg and Saks in [GS]. Only slight modification is needed to yield a countably compact space whose square is pseudocompact but not countably compact. Similar results can be obtained from the example given by Comfort in [C].

REFERENCES


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