AN INEQUALITY FOR FUNCTIONS OF EXPONENTIAL TYPE NOT VANISHING IN A HALF-PLANE

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Abstract. Let \( f(z) \) be an entire function of order 1, type \( \tau \) having no zero in \( \text{Im } z < 0 \). If \( h_f(-\pi/2) = \tau, \ h_f(\pi/2) < 0 \) then it is known that \( \sup_{-\infty < x < \infty} |f'(x)| > (\tau/2) \sup_{-\infty < x < \infty} |f(x)| \). In this paper we consider the case when \( f(z) \) has no zero in \( \text{Im } z < k, k < 0 \) and obtain a sharp result.

1. If \( f(z) \) is an entire function of exponential type \( \tau \) and \( |f(x)| < M \) for real \( x \), then according to a well-known theorem due to S. N. Bernstein [1, p. 206]

\[
|f'(x)| < Mr, \quad -\infty < x < \infty.
\]

If \( h_f(\pi/2) = 0 \),

\[
h_f(\theta) = \limsup_{r \to \infty} \frac{\log|f(re^{i\theta})|}{r}
\]

is the indicator function of \( f(z) \), and \( f(x + iy) \neq 0 \) for \( y > 0 \), then it has been proved by Boas [2] that (1.1) can be replaced by

\[
|f'(x)| \leq M\tau/2, \quad -\infty < x < \infty.
\]

This result of Boas is in fact a generalization of the Erdös conjecture proved by Lax [4] because the class of asymmetric entire functions of exponential type \( \tau \) includes all functions \( p(e^{iz}) \) where \( p(z) \) is a polynomial of degree \( n < [\tau] \) and \( p(z) \neq 0 \) in \( |z| < 1 \).

For polynomials having all their zeros in \( |z| < 1 \), we have the following result due to Turán [6].

**Theorem A.** If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|.
\]

As a generalization of Theorem A, an inequality analogous to (1.3) for entire functions of order 1, and type \( \tau \) has been obtained by Rahman [5] and for polynomials having all its zeros in \( |z| < K, K > 0 \) by Govil [3]. In this
paper we generalize the result due to Rahman [5] and due to Govil [3] (in the case $K > 1$) and prove the following.

**Theorem.** Let $f(z)$ be an entire function of order 1, type $\tau$ having all its zeros in $\text{Im} \, z > k$, $k < 0$. If $h_f(\pi/2) < 0$, $h_f(-\pi/2) = \tau$ then
\[
\sup_{-\infty < x < \infty} |f'(x)| > \frac{\tau}{1 + \exp(\tau|k|)} \sup_{-\infty < x < \infty} |f(x)|.
\]

The result is best possible with equality for the function
\[
f(z) = \left(\frac{e^{iz} - e^{-i\tau}}{1 + e^{-\tau k}}\right).
\]

2. For the proof of the theorem, we need the following lemmas.

**Lemma 1.** If $f(z)$ is an entire function of exponential type $\tau$ and $|f(x)| \leq M$, $-\infty < x < \infty$, then
\[
|f(x + iy)| \leq Me^{\tau|y|}, \quad -\infty < x < \infty, \quad -\infty < y < \infty.
\]

Lemma 1 is a simple consequence of the Phragmén-Lindelöf principle and follows immediately from a result due to Pólya and Szegö (see [1, p. 82, Theorem 6.2.4]).

**Lemma 2.** Let $f(z)$ be an entire function of order 1, type $\tau$, $h_f(\pi/2) < 0$, $|f(x)| \leq M$, $-\infty < x < \infty$, and let $g(z) = e^{iz}\text{con}\{f(z)\}$, where $\text{con}\{f(z)\}$ denotes the conjugate of $f(z)$. Then type $g \leq \tau$.

**Proof of Lemma 2.** If $g(z) = e^{iz}\text{con}\{f(z)\}$ is an entire function of order less than 1, then obviously type $g \leq \tau$, hence it is sufficient to prove the result when $g(z)$ is of order 1.

If $z = re^{i\theta}$ is a point of the upper half-plane, then
\[
|g(re^{i\theta})| = e^{-\tau \sin \theta} |f(re^{-i\theta})|,
\]
which gives by Lemma 1,
\[
|g(re^{i\theta})| \leq e^{-\tau \sin \theta} e^{\tau \sin \theta} \sup_{-\infty < x < \infty} |f(x)|
\]
\[
= \sup_{-\infty < x < \infty} |f(x)|.
\]

(2.2)

If $z = re^{i\theta}$ lies in the lower half-plane, the point $z = re^{-i\theta}$ will lie in the upper half-plane and since $h_f(\pi/2) < 0$, hence it follows by a result due to Pólya and Szegö (see [1, p. 82, Theorem 6.2.4]) that
\[
|g(re^{i\theta})| = e^{-\tau \sin \theta} |f(re^{-i\theta})|
\]
\[
\leq e^{-\tau \sin \theta} \sup_{-\infty < x < \infty} |f(x)|
\]
\[
\leq e^{\tau \tau} \sup_{-\infty < x < \infty} |f(x)|.
\]

(2.3)

On combining (2.2) and (2.3), we get
\[ |g(re^{i\theta})| \leq e^{r\tau} \sup_{-\infty < x < \infty} |f(x)|, \quad 0 \leq \theta < 2\pi, \]

which gives that type \( g \leq \tau = \text{type } f \), and Lemma 2 follows.

**Lemma 3.** If \( f(z) \) is an entire function of order 1, type \( \tau \) such that \( h_f(-\pi/2) = \tau, h_f(\pi/2) < 0 \), \( f(z) \) has all its zeros in \( \text{Im } z > k, k < 0 \), then

\[ (2.4) \sup_{-\infty < x < \infty} |g'(x)| \leq e^{r|x|} \sup_{-\infty < x < \infty} |f'(x)|, \]

where as in Lemma 2, \( g(z) \) stands for \( e^{iz\text{con}(f(z))} \) and \( \text{con}(f(\bar{z})) \) for the conjugate of \( f(\bar{z}) \).

**Proof of Lemma 3.** Let \( F(z) = f(z + ik) \) and \( G(z) = e^{iz\text{con}(F(\bar{z}))} = e^{-z}g(z - ik) \), where \( \text{con}(F(\bar{z})) \) denotes the conjugate of \( F(\bar{z}) \). Since \( f(z) \) has all its zeros in \( \text{Im } z > k, k < 0 \), \( h_f(-\pi/2) = \tau, h_f(\pi/2) < 0 \), the function \( F(z) \) is an entire function of order 1, type \( \tau \), has no zero in \( \text{Im } z < 0 \), \( h_{F}(\pi/2) = \tau \) and \( h_{F}(-\pi/2) < 0 \). Therefore the function \( F(z) \) belongs to the class \( P \). Further \( F_1(z) = e^{iz/2\text{con}(F(\bar{z}))} \) is an entire function of exponential type having no zero in \( \text{Im } z > 0 \) and satisfying \( h_{F}(\pi/2) > h_{F}(-\pi/2) \). Hence applying a result due to Levin (see [1, p. 129, Theorem 7.8.1]) to the function \( F_1(z) \), we get

\[ |e^{iz/2\text{con}(F(z))}| > |e^{iz/2\text{con}(F(z))}| \quad \text{for } \text{Im } z > 0, \]

which implies

\[ |e^{iz/2\text{con}(F(z))}| > |e^{iz/2\text{con}(F(z))}| \quad \text{for } \text{Im } z < 0, \]

and which implies

\[ |F(z)e^{-iz/2}| > |e^{iz/2\text{con}(F(z))}| \quad \text{for } \text{Im } z < 0. \]

Thus

\[ |F(z)| > |e^{iz\text{con}(F(z))}| \quad \text{for } \text{Im } z < 0. \]

Since \( G(z) = e^{iz\text{con}(F(\bar{z}))} \), we get

\[ (2.5) \quad |F(z)| > |G(z)| \quad \text{for } \text{Im } z < 0. \]

For \( k < 0 \), let \( F_k(z) \) denote the function \( F(z + ik) \) and \( G_k(z) \) the function \( G(z + ik) \). Then the function \( F_k(z) \) is an entire function of order 1, type \( \tau \). Also by Lemma 2, the function \( G_k(z) \) is an entire function of exponential type \( < \tau \). Since \( F(z) \) has no zero in \( \text{Im } z < 0 \), therefore \( F_k(z) \) has no zero in \( \text{Im } z < -k \), and hence no zero in \( \text{Im } z < 0 \), because \( k < 0 \). Further because \( h_{F_k}(-\pi/2) = h_{F}(-\pi/2) = \tau, h_{F_k}(\pi/2) = h_{F}(\pi/2) < 0 \), we get \( h_{F_k}(-\pi/2) > h_{F_k}(\pi/2) \) and therefore \( F_k(z) \) belongs to the class \( P \). Thus \( G_k(z) \) is an entire function of exponential type \( < \tau \) and \( F_k(z) \) an entire function of class \( P \), order 1 and type \( \tau \). Also by (2.5) we have \( |G_k(x)| < |F_k(x)|, \quad -\infty < x < \infty \), hence applying a result due to Levin (see [1, p. 226, Theorem 11.7.2]) and the fact that differentiation is a \( B \)-operator, we get \( |G_k(x)| < |F_k'(x)|, \quad -\infty < x < \infty \), which implies
(2.6) \[ |G'(x + ik)| < |F'(x + ik)|, \quad -\infty < x < \infty, k < 0. \]

Since \[ |F'(x + ik)| = |f'(x + 2ik)|, \quad \text{and} \quad |G'(x + ik)| = e^{-\tau k} |g'(x)|, \] (2.6) gives,

(2.7) \[ |g'(x)| \leq e^{\tau k} |f'(x + 2ik)|, \quad -\infty < x < \infty. \]

Lastly applying the inequality (2.1) to \[ |f'(x + 2ik)| \] and combining it with (2.7) we get

\[ |g'(x)| \leq e^{\tau |k|} \sup_{-\infty < x < \infty} |f'(x)|, \]

from which the lemma follows.

**Lemma 4.** If \( f(z) \) is an entire function of exponential type \( \tau \), then

\[ \sup_{-\infty < x < \infty} |f'(x)| + \sup_{-\infty < x < \infty} |g'(x)| > \tau \sup_{-\infty < x < \infty} |f(x)|, \]

where \( g(z) \) is the same as defined in Lemma 3.

**Proof of Lemma 4.** From the definition of \( g(z) \) it follows that on real axes,

\[ |g'(x)| = |e^{ix} f'(x) + ire^{ix} f(x)| \]

\[ > \tau |f(x)| - |f'(x)|. \]

Thus for \( -\infty < x < \infty \), \[ |f'(x)| + |g'(x)| > \tau |f(x)|, \]

which gives

\[ \sup_{-\infty < x < \infty} |f'(x)| + \sup_{-\infty < x < \infty} |g'(x)| > \tau \sup_{-\infty < x < \infty} |f(x)|, \]

and Lemma 4 is proved.

3. **Proof of the Theorem.** We have by Lemma 3,

(3.1) \[ \sup_{-\infty < x < \infty} |g'(x)| \leq e^{\tau |k|} \sup_{-\infty < x < \infty} |f'(x)|. \]

Also by Lemma 4,

(3.2) \[ \sup_{-\infty < x < \infty} |f'(x)| + \sup_{-\infty < x < \infty} |g'(x)| \geq \tau \sup_{-\infty < x < \infty} |f(x)|. \]

Combining (3.1) and (3.2) we get

\[ \sup_{-\infty < x < \infty} |f'(x)| + e^{\tau |k|} \sup_{-\infty < x < \infty} |f'(x)| \geq \tau \sup_{-\infty < x < \infty} |f(x)|, \]

which implies

\[ \sup_{-\infty < x < \infty} |f'(x)| \geq \frac{\tau}{1 + e^{\tau |k|}} \sup_{-\infty < x < \infty} |f(x)|. \]

This completes the proof of the theorem.

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**References**


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