PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 65, Number 2, August 1977

A NOTE ON COMPARISON THEOREMS

SHOSHANA ABRAMOVICH

ABSTRACT. Comparison theorems for solutions, for focal points and conjugate points of second order linear differential equations are given in this paper.

Comparison theorems inspired by theorems of Leighton and Woo Kian Ke [4], Leighton [5], Guggenheimer [2] and Walter [6] are given.

The first focal point $f(a)$ of an equation $x'' + p(t)x = 0$ is the first zero $> a$ of a derivative of a nontrivial solution of the equation which itself vanishes at a. Similarly the second focal point $g(a)$ is defined as the first zero $> a$ of a solution of an equation whose derivate vanishes at a. The first conjugate point $c(a)$ is the first zero $> a$ of a nontrivial solution of the equation that vanishes at a.

The purpose of this note is to prove comparison theorems for solutions of the equations

$$u'' + pu = 0, \tag{1}$$

$$y'' + qy = 0, \tag{2}$$

and comparison theorems for focal points and conjugate points. We assume that p and q are positive and continuous in the interval we are concerned with.

THEOREM 1. *Suppose that* $u(x)$ *and* $y(x)$ *are positive solutions of* (1) *and* (2) *in the interval* (a, b). *If,*

(i) $u(a) = y(a) = 0$, $u'(a) = y'(a) = 1$, $u'(b) = y'(b) = 0$;

(ii) *there exists* $d \in (a, b)$ *so that* $p \geqslant q$ *for* $a \leqslant x < d$, *and* $\int_x^b (q - p)\,dt \geqslant 0$ *for* $d < x \leqslant b$, *then*

$$u(x) \leqslant y(x), \qquad a < x \leqslant b, \tag{3}$$

$$f_p'(a) \geqslant f_q'(a). \tag{4}$$

PROOF. Since $\int_x^b (q - p)\,dt \geqslant 0$, $d \leqslant x \leqslant b$, it follows that

$$\int_x^b (q - p)uy\,dt \geqslant 0, \qquad d \leqslant x \leqslant b \tag{5}$$

(see [3, Theorem 399]).

Since

Received by the editors October 26, 1976.
AMS (MOS) subject classifications (1970). Primary 34C10.

© American Mathematical Society 1977

(6) $$y'u - u'y = \int_x^b (q - p)uy\, dt = \int_a^x (p - q)uy\, dt,$$

by (ii) and (5) $y'u - u'y \geqslant 0$; hence $(y/u)' \geqslant 0$, therefore $u'(a) = y'(a) = 1$ imply (3).

It follows from (ii) that $p(b) \leqslant q(b)$, so we get (4) from $f'_p(a) = (1/p(b))(u'^2(a)/u^2(b))$ [1, p. 121 (13.7)].

It is evident from this proof that if the inequalities in (ii) are strict in any subinterval of (a, d) or of (d, b), then also inequalities (3) and (4) are strict.

Using the same methods as in the above theorem together with those of [4], we get the following theorem.

THEOREM 2. *Let $u(x)$ and $y(x)$ be positive solutions of* (1) *and* (2). *If,*

(i) $y'(a) = u'(a) = 0, y(a) = u(a) = 1, y(b) = u(b) = 0,$

(ii) *there exists $d \in (a, b)$ so that $\int_a^x (p - q)dt \leqslant 0$, $a < x < d$, and $p(x) \geqslant q(x)$, $d < x < b$, $p(b) = q(b)$; then*

(7) $$u(x) \geqslant y(x), a < x < b,$$

(8) $$|u'(b)| \geqslant |y'(b)|,$$

and

(9) $$g'_p(a) \leqslant g'_q(a).$$

If the inequalities in (ii) *are strict in any subinterval of* (a, d) *or of* (d, b) *then the inequalities* (7), (8) *and* (9) *are strict. Inequality* (9) *follows from the derivation formulae* $g'(a) = p(b)u^2(a)/u'^2(b)$ [5], [1, *p.* 121 (13.8)].

The last theorem is about conjugate points.

THEOREM 3. *Suppose that $u(x)$ and $y(x)$ are positive solutions of* (1) *and* (2) *in* $(-a, a)$. *If,*

(i) $y(-a) = y(a) = u(-a) = u(a) = 0, y'(-a) = u'(-a),$

(ii) $p(-x) \geqslant p(x), q(-x) \geqslant q(x), a < x \leqslant a,$

(iii) $q(x) \geqslant p(x), -a \leqslant x \leqslant 0,$

(iv) *there exists a point d so that*

$$\int_{-x}^x (q - p)dt \geqslant 0, \qquad 0 \leqslant x \leqslant d,$$

and $q(x) \leqslant p(x)$, $d < x \leqslant a$, then,

(10) $$y \leqslant u, \qquad -a < x < a,$$

(11) $$|y'(a)| \leqslant |u'(a)|,$$

and

(12) $$c'_p(-a) \leqslant c'_q(-a).$$

If the inequalities in (iii) *and* (iv) *are strict in any subinterval of* $(-a, 0)$ *or of* $(0, d)$ *or* (d, a) *then the inequalities* (10), (11) *and* (12) *are strict too.*

PROOF. It can be easily verified that if (ii) holds then

(13) $$y(-x) \geqslant y(x), \quad u(-x) \geqslant u(x), \qquad 0 \leqslant x \leqslant a.$$

Since

$$w = y'u - u'y = \int_{-a}^{x} (p - q)uy\, dt = \int_{x}^{a} (q - p)uy\, dt$$

it follows that $w \leqslant 0$ in the intervals $[-a, 0]$ and $[d, a]$.

Now, if $0 \leqslant x \leqslant d$, then,

$$\begin{aligned}
w = \int_{-a}^{x} (p - q)uy\, dt &= \int_{-a}^{-x} (p - q)uy\, dt \\
&\quad + \int_{-x}^{0} (p - q)uy\, dt + \int_{0}^{x} (p - q)uy\, dt \\
&\leqslant \int_{-a}^{-x} (p - q)uy\, dt + \int_{-z}^{-x} (p - q)uy\, dt \\
&\quad + \int_{-x}^{0} (p - q)u(-t)y(-t)\, dt + \int_{0}^{x} (p - q)uy\, dt \\
&= \int_{-a}^{-x} (p - q)uy\, dt + \int_{-x}^{x} (p - q)u^*y^*\, dt \leqslant 0,
\end{aligned} \tag{14}$$

where

$$u^*y^* = \begin{cases} u(x)y(x), & 0 \leqslant x \leqslant a, \\ u(-x)y(-x), & -a \leqslant x \leqslant 0. \end{cases}$$

By (13) and (iii) we get the first inequality in (14). As $y(x)$ and $u(x)$ decrease in $[0, a]$ by (iv), using the simple corollary of [3, Theorem 399] that $\int_{-x}^{x}(p - q)u^*y^*\, dt \leqslant 0$, the second inequality sign in (14) is established. This completes the proof that $w \leqslant 0$, $-a \leqslant x \leqslant a$.

Continuing the same methods as the previous theorems and using the formulae $c'(-a) = u'^2(-a)/u'^2(a)$ [1], [4], [5], we get the assertions of the theorem.

References

1. O. Borůvka, *Linear differential transformations of the second order*, English Univ. Press, London, 1971.

2. H. Guggenheimer, *A comparison theorem*, Proc. Amer. Math. Soc. **55** (1976), 310–312.

3. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, London and New York, 1952.

4. W. Leighton and W. Oo Kian Ke, *A comparison theorem*, Proc. Amer. Math. Soc. **28** (1971), 185–188.

5. W. Leighton, *Principal quadratic functionals*, Trans. Amer. Math. Soc. **68** (1949), 253–274.

6. W. Walter, *A comparison theorem and an inequality of Redheffer*, Applicable Anal. **2** (1971), 111–113.

Department of Mathematics, University of Haifa, Haifa, Israel