

## A SHORT SHORT PROOF OF THE CARTWRIGHT-LITTLEWOOD THEOREM

MORTON BROWN

**ABSTRACT.** Each orientation preserving homeomorphism of the plane that is invariant on a nonseparating bounded continuum has a fixed point on the continuum.

**Introduction.** In 1951 Cartwright and Littlewood [1] proved that an orientation preserving homeomorphism  $h$  of the Euclidean plane  $R^2$  that leaves a compact nonseparating continuum  $C$  invariant leaves a point of  $C$  fixed. In 1954 Hamilton [2] gave a short proof employing the result of Brouwer [3]: An orientation preserving homeomorphism of  $R^2$  that leaves a bounded set invariant has a fixed point (possibly not in the set). We give another short proof also based on Brouwer's theorem.

**PROOF OF THEOREM.** Let  $F$  be the fixed point set of  $h$ . We shall derive a contradiction from the hypothesis that  $F \cap C = \emptyset$ . Let  $U$  be the component of  $R^2 - F$  that contains  $C$ . Then  $h(U) = U$ . Let  $\tilde{U}$  be the universal cover of  $U$ . ( $\tilde{U}$  then is homeomorphic to  $R^2$ .) Then  $C$  lifts to disjoint homeomorphic copies of  $C$  in  $\tilde{U}$ . (The hypotheses on  $C$  insure that there exists a disk  $D$  such that  $C \subset D \subset U$ , and  $\tilde{D}$  being simply connected lifts to disjoint homeomorphic copies of  $\tilde{D}$ .) Let  $\tilde{C}$  be one of these copies of  $C$  and let  $\tilde{h}: \tilde{U} \rightarrow \tilde{U}$  be a homeomorphism covering  $h|U: U \rightarrow U$  and invariant on  $\tilde{C}$ . Since  $h|U$  is orientation preserving, so is  $\tilde{h}$ . Since  $h|U$  has no fixed points neither does  $\tilde{h}$ . Thus  $\tilde{h}$  is an orientation preserving fixed point free homeomorphism of  $R^2$  that is invariant on  $\tilde{C}$ . This contradicts Brouwer's theorem.

**REMARK.** Brouwer's theorem is false if  $h$  is orientation reversing and neither this proof, Hamilton's, nor Cartwright-Littlewood's works. That the theorem in question remains true is a difficult result recently established by H. Bell [4].

### REFERENCES

1. M. L. Cartwright and J. C. Littlewood, *Some fixed point theorems*, Ann. of Math. **54** (1951), 1-37.
2. O. H. Hamilton, *A short proof of the Cartwright Littlewood fixed point theorem*, Canad. J. Math. **6** (1954), 522-523.
3. L. E. J. Brouwer, *Beweis des Ebenen Translationssatzes*, Math. Ann. **72** (1912), 36-54.
4. Harold Bell, *A fixed point theorem for planar homeomorphisms*, Bull. Amer. Math. Soc. **82** (1976), 778-780.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109

Received by the editors January 3, 1977.

AMS (MOS) subject classifications (1970). Primary 57A05, 55C20, 54H25.

Key words and phrases. Fixed point, homeomorphism, Cartwright-Littlewood.

© American Mathematical Society 1977